

Dibaryons with two strange quarks and one heavy flavor in a constituent quark model

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(Dated: June 10, 2016)

We investigate the symmetry property and the stability of dibaryons containing two strange quarks and one heavy flavor with $I = \frac{1}{2}$. We construct the wave function of the dibaryon in two ways. First, we directly construct the color and spin state of the dibaryon starting from the four possible SU(3) flavor state. Second, we consider the states composed of five light quarks, and then construct the wave function of the dibaryon by adding one heavy quark. The stability of the dibaryon against the strong decay into two baryons is discussed by using variational method in a constituent quark model with confining and hyperfine potential. We find that for all configurations with $S=0,1,2$, the ground states of the dibaryons are the sum of two baryons, and there are no compact bound state that is stable against the strong decay.

PACS numbers: 14.40.Rt, 24.10.Pa, 25.75.Dw

I. INTRODUCTION

Investigating the stability of multi-quark hadrons have been pursued in various models after Jaffe suggested the possible existence of such particles in QCD [1–3]. The observation of many charmonium-like states, such as $X(3872)$, $Z_c(3900)$ and $Z^+(4430)$, and of heavy pentaquark states [4] revived great interest in the studies of multi-quark hadrons and/or of molecular bound states containing heavy quark hadrons. Additionally, a new particle, called $X(5568)$ was recently observed by the D0 collaboration in the $B_s^0 \pi^\pm$ invariant mass spectrum with 5.1σ significance [5]. $X(5568)$ may be the first observed tetraquark which has four different flavors: up, down, strange and bottom. If all the flavors are different, for any typical two body interaction, one can always find the most attractive combination so that one has advantage to form a bound multi-quark state.

The stability of the dibaryons with heavy quarks were studied already in several models; those based on chromomagnetic models [6, 7] and chiral constituent quark model [8, 9]. Furthermore, Huang et al. studied H -like dibaryon states containing heavy quarks instead of strange quarks within the framework of the quark delocalization color screening model [10]. Dibaryons within the diquark models with heavy quarks are also considered [11].

Most of the models studying the possible existence of dibaryons are looking at the most attractive color spin interaction channel [1–3]. For example, for the H dibaryon, the attraction in the color spin interaction is larger than those coming from two Λ 's, which is the most attractive two baryon channel that the dibaryon can decay. However, it should be noted that whether such attraction really leads to a stable compact dibaryon states is de-

termined by whether the attraction is strong enough to overcome the extra repulsion coming from bringing all the quarks together into a compact configuration. As we will discuss later, the magnitude of each effect depends on the masses of the quarks involved that can only be systematically studied within a complete model that consistently treats the kinetic terms and the interaction terms within one framework.

In this work, to investigate the subtle interplay between the two competing effects, in a simple but consistent model, we will study the stability of $uudssQ$ dibaryon using variational method in a constituent quark model. This is a generalization of H -dibaryon to include one heavy quark so that it contains the most attractive color-spin interaction channel but at the same time, reduced kinetic energy from combining six quarks in a compact configuration. In particular, we focus our attention on $I = \frac{1}{2}$ because the states with lowest isospin are most attractive bound for a given quark system [6].

Moreover, we will demonstrate how to consistently construct the color spin flavor wave functions that contains the $uudssQ$ quarks. There are two ways of constructing the wave function of dibaryon with one heavy quark. First, we can directly construct the color and spin wave function of dibaryon starting from the four possible SU(3) flavor state. Or we can consider the color and spin state of q^5 and then construct the wave function of dibaryon by adding one heavy quark. We show the two approaches lead to identical wave functions, showing the consistency of our approach. Technically, the second approach is more convenient to obtain the wave function compared to using the first approach, because the former utilizes Clebsch-Gordan coefficients of S_6 while the second approach uses that of S_5 .

This paper is organized as follows. We first present the Hamiltonian and calculate the masses of baryons to determine the fitting parameters of the model in Sec.II. In Sec.III, we explain why we choose the dibaryon with one heavy flavor in terms of the relation between hyperfine potential and the stability condition. In Sec.IV, we construct the spatial wave function of the dibaryon. In

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TABLE I: Parameters fitted to the experimental baryon masses using the variational method with a single Gaussian. The respective units are given in the third row.

κ	κ'	a_0	D	α	β	m_q	m_s	m_c	m_b
0.59	0.5	5.386	0.960	2.6	0.552	0.343	0.632	1.93	5.3
		GeV ⁻²	GeV	(fm) ⁻¹		GeV	GeV	GeV	GeV

Sec.V, we classify q^5 with SU(3) flavor symmetry and construct the color and spin wave function of dibaryon using the first method. In Sec.VI, we construct the color and spin wave function of dibaryon using the second method. In Sec.VII, we calculate the wave function of dibaryon and show that their results are the same in both methods. In Sec.VIII, we represent the numerical results obtained from the variational method, and finally summarize the results in Sec.IX. The appendices include some details of the calculations.

II. HAMILTONIAN

We take a nonrelativistic Hamiltonian with the confinement and hyperfine potential given by

$$H = \sum_{i=1}^6 (m_i + \frac{\mathbf{p}_i^2}{2m_i}) - \frac{3}{16} \sum_{i<j} \lambda_i^c \lambda_j^c (V_{ij}^C + V_{ij}^{SS}), \quad (1)$$

where m_i 's are the quark masses, and $\lambda_i^c/2$ are the color operator of the i 'th quark for the color SU(3), and V_{ij}^C and V_{ij}^{SS} are the confinement and hyperfine potential, respectively.

$$V_{ij}^C = -\frac{\kappa}{r_{ij}} + \frac{r_{ij}}{a_0} - D. \quad (2)$$

The hyperfine term which effectively splits the multiplets of baryon with respect to spin is expressed as

$$V_{ij}^{SS} = \frac{\hbar^2 c^2 \kappa'}{m_i m_j c^4} \frac{1}{(r_{0ij})^2 r_{ij}} e^{-(r_{ij})^2/(r_{0ij})^2} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j. \quad (3)$$

Here, r_{ij} is the distance between interquarks, $|\mathbf{r}_i - \mathbf{r}_j|$, and r_{0ij} are chosen to depend on the masses of interquarks given by

$$r_{0ij} = 1/(\alpha + \beta \frac{m_i m_j}{m_i + m_j}). \quad (4)$$

We choose to keep the isospin symmetry by requiring that $m_u = m_d$. In the Hamiltonian, the parameters have been chosen so that the fitted mass of baryons are comparable with those of experiments.

When we calculate the expectation value of the potential terms for baryon with certain symmetry, it is convenient to introduce the following three Jacobian coordinates. Then it reduces our problem to the two body-system in the center of mass frame :

TABLE II: This table shows the masses of baryons obtained from the variational method. The third row indicates the experimental data. (unit : GeV)

(I,S)	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{2})$	$(0, \frac{1}{2})$	$(1, \frac{1}{2})$	$(1, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{2})$
	N, P	Δ	Λ	Σ	Σ^*	Ξ	Ξ^*
Mass	0.977	1.23	1.12	1.2	1.38	1.324	1.52
Exp	0.938	1.232	1.115	1.189	1.382	1.315	1.532

(I,S)	$(0, \frac{1}{2})$	$(1, \frac{1}{2})$	$(1, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{2})$	$(0, \frac{1}{2})$	$(0, \frac{3}{2})$
	Λ_c	Σ_c	Σ_c^*	Ξ_c	Ξ_c^*	Ω_c	Ω_c^*
Mass	2.285	2.45	2.526	2.476	2.649	2.687	2.763
Exp	2.286	2.453	2.518	2.468	2.646	2.695	2.766

(I,S)	$(0, \frac{1}{2})$	$(1, \frac{1}{2})$	$(1, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{2})$	$(0, \frac{1}{2})$	$(0, \frac{3}{2})$
	Λ_b	Σ_b	Σ_b^*	Ξ_b	Ξ_b^*	Ω_b	Ω_b^*
Mass	5.608	5.809	5.839	5.787	5.95	6.019	6.053
Exp	5.619	5.811	5.832	5.792	5.949	6.048	

- Coordinate I :

$$\begin{aligned} \mathbf{x}_1 &= \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \\ \mathbf{x}_2 &= \frac{m_1 + m_2}{\sqrt{2m_1^2 + 2m_2^2 + 2m_1m_2}} \left(\frac{m_1}{m_1 + m_2} \mathbf{r}_1 \right. \\ &\quad \left. + \frac{m_2}{m_1 + m_2} \mathbf{r}_2 - \mathbf{r}_3 \right). \end{aligned} \quad (5)$$

- Coordinate II :

$$\begin{aligned} \mathbf{y}_1 &= \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_3), \\ \mathbf{y}_2 &= \frac{m_2 + m_3}{\sqrt{2m_2^2 + 2m_3^2 + 2m_2m_3}} \left(\frac{m_2}{m_2 + m_3} \mathbf{r}_2 \right. \\ &\quad \left. + \frac{m_3}{m_2 + m_3} \mathbf{r}_3 - \mathbf{r}_1 \right). \end{aligned} \quad (6)$$

- Coordinate III :

$$\begin{aligned} \mathbf{z}_1 &= \frac{1}{\sqrt{2}}(\mathbf{r}_3 - \mathbf{r}_1), \\ \mathbf{z}_2 &= \frac{m_1 + m_3}{\sqrt{2m_1^2 + 2m_3^2 + 2m_1m_3}} \left(\frac{m_1}{m_1 + m_3} \mathbf{r}_1 \right. \\ &\quad \left. + \frac{m_3}{m_1 + m_3} \mathbf{r}_3 - \mathbf{r}_2 \right). \end{aligned} \quad (7)$$

By using simple Gaussian function, we calculate the baryon masses containing charm or bottom quark. The method to calculate the color, flavor and spin basis was explained in Ref. [12].

III. COLOR-SPIN INTERACTION AND THE STABILITY CONDITION

It is well known that color-spin interaction is an important factor in investigating the stability of multi-quark system. In SU(3) flavor symmetry, there is a simple formula [13] from which one can easily calculate the expectation value of hyperfine potential.

$$H_{SS} = - \sum_{i < j}^N \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j$$

$$= N(N-10) + \frac{4}{3}S(S+1) + 2C_C + 4C_F \quad (8)$$

where $C_F = \frac{1}{4}\lambda^F\lambda^F$. For the flavor singlet H-dibaryon, $H_{SS} = -24$, and for Λ , $H_{SS} = -8$. Hence, H-dibaryon is more attractive than $\Lambda\Lambda$ system in terms of color-spin interaction. This is the basis for a possible stable H-dibaryon.

At the same time, it is interesting to point out that we can split the dibaryon into five quarks and one quark system. By using Eq. (8) we can calculate the expectation value of hyperfine potential of five quarks in H-dibaryon. We represent the expectation values of hyperfine potential for H-dibaryon with flavor singlet and $\Lambda\Lambda$ in Table III. In that case, the flavor and color state of five quarks are anti-triplet and $C_C = C_F = \frac{4}{3}$ for anti-triplet state so it leads to $H_{SS} = -16$ which is the same as $H_{SS}^{\Lambda\Lambda}$. So it shows that the interaction between sixth quark and the other quarks give more attractive effect than $\Lambda\Lambda$ system and agrees with our recent work [14]. Unfortunately, H-dibaryon is not stable in our model when we consider the Hamiltonian as the repulsion coming from kinetic energy and confinement potential dominates over the attraction coming from hyperfine potential as we bring six quarks to compact configuration.

If we replace the sixth quark with heavy quark, then the situation becomes more subtle. In infinite heavy quark mass limit, the contribution from sixth quark becomes zero because hyperfine potential has $1/m_Q$ factor. And in that case, the expectation value of hyperfine potential is the same as that of Λ and diquark system. However, heavy quark mass is not infinite, so $1/m_Q$ factor weakens the attractive effect, but it will also reduce the kinetic energy if it doesn't change the interquark distances. So we can consider the dibaryon with heavy flavor and it may lead more chance to form the stable state than H-dibaryon. It should be noted that anti-triplet flavor state is the most attractive color-spin interaction when we consider five quarks only.

IV. SPATIAL FUNCTION

In order to construct an antisymmetric wave function of the dibaryon, we choose the spatial function to be symmetric such that the rest of the wave function represented

TABLE III: The expectation value of $-\sum_{i < j} \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$ for H-dibaryon with flavor singlet and $\Lambda\Lambda$.

$-\sum_{i < j} \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$	$i < j = 1-5$	$i = 1-5, j = 6$
H-dibaryon, F^1	-16	-8

$-\sum_{i < j} \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$	$i < j = 1-3$	$i, j = 4-5$	$i=4-5, j=6$
$\Lambda\Lambda$	-8	-8	0

by color \otimes flavor \otimes spin should be antisymmetric. Here, we calculate in the flavor SU(3) breaking case and fix the position of each quarks on $u(1)u(2)d(3)s(4)s(5)Q(6)$. So our wave function should have the specific symmetry property which is antisymmetric among 1,2 and 3, and at the same time antisymmetric between 4 and 5. And among various Jacobi coordinates, we choose the baryon-baryon configuration because it is convenient to investigate the strong decay mode.

$$\sum_{i=1}^6 \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - \frac{1}{2} M \dot{\mathbf{r}}_{CM}^2 = \sum_{i=1}^5 \frac{1}{2} M_i \dot{\mathbf{x}}_i^2$$

where $M_1 = M_2 = m$, $M_3 = m_s$, $M_4 = \frac{3m_s m_Q}{2m_s + m_Q}$,

$$M_5 = \frac{2m(5m_s^2 + 2m_s m_Q + 2m_Q^2)}{(3m + 2m_s + m_Q)(2m_s + m_Q)} \quad (9)$$

The Jacobian coordinates are given by

$$\begin{aligned} \mathbf{x}_1 &= \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \\ \mathbf{x}_2 &= \sqrt{\frac{2}{3}}(\frac{1}{2}\mathbf{r}_1 + \frac{1}{2}\mathbf{r}_2 - \mathbf{r}_3), \\ \mathbf{x}_3 &= \frac{1}{\sqrt{2}}(\mathbf{r}_4 - \mathbf{r}_5), \\ \mathbf{x}_4 &= \sqrt{\frac{2}{3}}(\frac{1}{2}\mathbf{r}_4 + \frac{1}{2}\mathbf{r}_5 - \mathbf{r}_6), \\ \mathbf{x}_5 &= \frac{\sqrt{3}(2m_s + m_Q)}{\sqrt{10m_s^2 + 4m_s m_Q + 4m_Q^2}}(\frac{1}{3}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2 + \frac{1}{3}\mathbf{r}_3 \\ &\quad - \frac{m_s}{2m_s + m_Q}\mathbf{r}_4 - \frac{m_s}{2m_s + m_Q}\mathbf{r}_5 - \frac{m_Q}{2m_s + m_Q}\mathbf{r}_6). \end{aligned} \quad (10)$$

Then, we can construct the spatial wave function of the dibaryon in a single Gaussian form that can accommodate the required symmetry property:

$$R = \exp[-a(x_1^2 + x_2^2) - bx_3^2 - cx_4^2 - dx_5^2], \quad (11)$$

where a, b, c and d are the variational parameters. The spatial function in Eq. (11) is symmetric among 1,2 and 3, and at the same time symmetric between 4 and 5. We will denote this symmetry property of the spatial function

by [123][45]6. Considering the dibaryon to be formed by bringing together a baryon composed of particle [123] and baryon composed of [45]6, one notes that the additional kinetic term will involve coordinate \mathbf{x}_5 with mass M_5 . Hence, for fixed \mathbf{x}_5 , the additional kinetic term becomes smaller when m_Q increases but only becomes zero when more than one quark becomes heavy as can be seen in Eq. (9). This additional kinetic term has to be smaller than the additional attraction coming from the color spin interaction for the dibaryon to form a stable compact state.

V. CLASSIFICATION OF q^5Q WITH $SU(3)$ FLAVOR SYMMETRY

In this section, we directly construct the color and spin wave function of the dibaryon from the four possible $SU(3)$ flavor state.

A. Flavor state of q^5

Here, we classify the flavor states in terms of $SU(3)_F$ symmetry and will break the flavor symmetry later. Since spatial wave function is symmetric, we have to construct the flavor, color and spin wave function to be antisymmetric. Under the general group $SU(18)_{CFS}$ totally antisymmetric multiplet of $[1^5]_{FCS}$ can be decomposed as

$$[1^5]_{FCS} = ([\bar{3}]_F, [420]_{CS}) \oplus ([6]_F, [336]_{CS}) \oplus ([\bar{15}]_F, [210]_{CS}) \oplus ([24]_F, [84]_{CS}) \oplus ([21]_F, [\bar{6}]_{CS}). \quad (12)$$

In this article, we consider only $I = \frac{1}{2}$, so that we exclude $[21]_F$ flavor state. Hence, there are four possible flavor states as follows.

$$[\bar{3}]_F = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, [6]_F = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}, [15]_F = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array},$$

$$[24]_F = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}.$$

For each flavor state, we can determine the possible Young tableau of color and spin state of q^5Q . According to group theory, for a given Young tableau, the fully antisymmetric state can be constructed by multiplying the Young tableau by its conjugate of the Young tableau, where the conjugate representation of a given Young tableau can be obtained by exchanging the row and column in the Young tableau. Additionally, the Young tableau of color and spin state that can contribute to the final state depends on the spin of the dibaryon. For a given spin state, the possible color and spin state can be obtained by taking the direct product of the color singlet dibaryon configuration to spin state and taking the conjugate, with the addition of the 6'th quark, of the fixed flavor state. We represent the possible flavor, color and spin state of q^5Q with $I = \frac{1}{2}$ for each spin states as follows.

- S=0 : 4 states

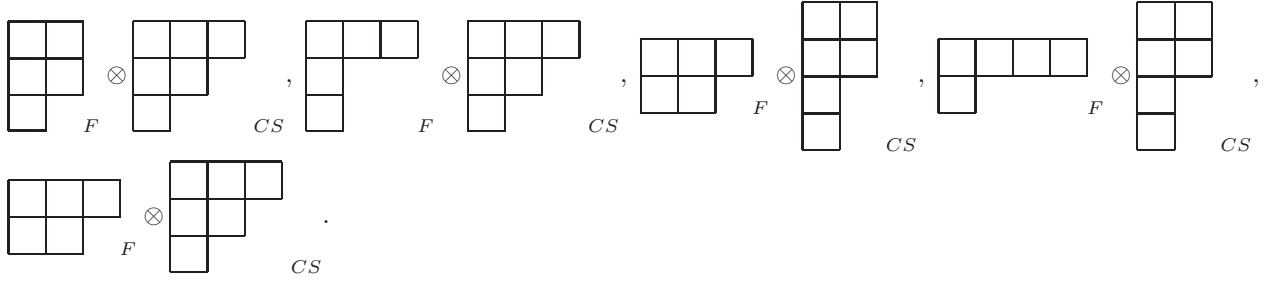
$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{CS}.$$

- S=1 : 8 states

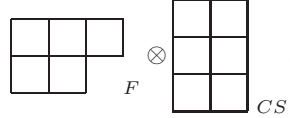
$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{CS},$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_{CS}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \square \\ \hline \square & & & \square \\ \hline \end{array}_F \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \square \\ \hline \square & & \square \\ \hline \end{array}_{CS}.$$

- S=2 : 5 states



- S=3 : 1 state



After $SU(3)_F$ breaking, there are only two flavor bases.

$$|F_1\rangle = \left(\begin{bmatrix} 1 & 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \end{bmatrix} \right), \quad |F_2\rangle = \left(\begin{bmatrix} 1 & 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \end{bmatrix} \right). \quad (13)$$

B. Flavor, color and spin state of $q^5 Q$

Here, we fix u and d quarks to be 1,2, and 3, and two s quarks to be 4 and 5. Since there is a flavor symmetry between strange quarks, the color and spin wave function should be antisymmetric between 4 and 5. The details to construct color and spin wave function which has specific symmetry property was explained in Ref. [14].

For example, in the case of $[3]_F$ and S=0 state, the Young tableau of color and spin state should be $[3,3]$. Furthermore, the color and spin state have to be antisymmetric between 4 and 5 because of their flavor symmetry. Hence, by using Young-Yamanouchi representation and permutation property, we can construct the flavor, color and spin wave function which has the required symmetry.

- S=0

$$\begin{aligned} \phi_1^{S=0} &= \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}_{CS} \right) \\ &\quad - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{bmatrix}_{CS} \right) \\ \phi_2^{S=0} &= \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 \end{bmatrix}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{bmatrix} 1 & 3 & 4 & 6 \\ 2 & 5 \end{bmatrix}_{CS} \right) \\ &\quad - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 \end{bmatrix}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{bmatrix} 1 & 2 & 4 & 6 \\ 3 & 5 \end{bmatrix}_{CS} \right) \end{aligned}$$

$$\begin{aligned} \phi_3^{S=0} &= \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}_{CS} \right) \\ &\quad - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}_{CS} \right) \end{aligned}$$

$$\phi_4^{S=0} = \frac{1}{\sqrt{2}} |F_1\rangle \otimes \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 5 \end{bmatrix}_{CS} - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 4 & 5 \end{bmatrix}_{CS} \quad (14)$$

- S=1

$$\begin{aligned} \phi_1^{S=1} &= \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 6 \end{bmatrix}_{CS} \right) \\ &\quad - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{1}{2} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{3}}{2} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{bmatrix}_{CS} \right) \\ \phi_2^{S=1} &= \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 \\ 6 \end{bmatrix}_{CS} \right) \\ &\quad - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 \\ 6 \end{bmatrix}_{CS} - \frac{\sqrt{5}}{\sqrt{8}} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{bmatrix}_{CS} \right) \end{aligned}$$

quarks first and then, we will add a heavy quark later. The totally antisymmetric multiplet of $[1^5]_{CFS}$ can be decomposed as

$$[1^5]_{CFS} = ([\bar{3}]_C, [420]_{FS}) \oplus ([6]_C, [336]_{FS}) \oplus ([\bar{15}]_C, [210]_{FS}) \oplus ([24]_C, [84]_{FS}) \oplus ([21]_C, [\bar{6}]_{FS}). \quad (17)$$

By using the Young tableau, we can find that the multiplets in the right hand side of Eq. (17) is fully antisymmetric.

$$\begin{aligned} ([\bar{3}]_C, [420]_{FS}) &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \\ ([6]_C, [336]_{FS}) &= \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \\ ([\bar{15}]_C, [210]_{FS}) &= \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \\ ([24]_C, [84]_{FS}) &= \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \\ ([21]_C, [6]_{FS}) &= \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}. \end{aligned}$$

Since the dibaryon is a color singlet, the color state of q^5 should be anti-triplet. So flavor and spin state of q^5 should be $[420]_{FS}$. The $[420]_{FS}$ multiplet can be decomposed as

$$[420]_{FS} = ([\bar{3}]_F, [2]_S) \oplus ([6]_F, [2]_S) \oplus ([\bar{15}]_F, [2]_S) \oplus ([21]_F, [2]_S) \oplus ([24]_F, [2]_S) \oplus ([\bar{3}]_F, [4]_S) \oplus ([6]_F, [4]_S) \oplus ([\bar{15}]_F, [4]_S) \oplus ([24]_F, [4]_S) \oplus ([\bar{15}]_F, [6]_S). \quad (18)$$

The corresponding Young tableau of each states are given as follows.

$$\begin{aligned} ([\bar{3}]_F, [2]_S) &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \\ ([6]_F, [2]_S) &= \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \end{aligned}$$

$$([\bar{15}]_F, [2]_S) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array},$$

$$([24]_F, [2]_S) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array},$$

$$([\bar{3}]_F, [4]_S) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$$

$$([6]_F, [4]_S) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$$

$$([\bar{15}]_F, [4]_S) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$$

$$([24]_F, [4]_S) = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array},$$

$$([\bar{15}]_F, [6]_S) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \end{array}.$$

In this article, we consider only $I = \frac{1}{2}$, so we exclude $[21]_F$ case. Therefore, there are four flavor states for each $S = \frac{1}{2}$ and $S = \frac{3}{2}$, and one state for $S = \frac{5}{2}$. Hence, as we can see in Figure 1, we can determine the number of possible states of q^5Q when adding one heavy quark. For the dibaryons, there can be four spin states $S=0,1,2,3$. From the above decomposition, for $S=0$ case, there are 4 possible states. For $S=1$, there are 8 possible states. For $S=2$, there are 5 possible states. For $S=3$, there is only one flavor that is $[\bar{15}]$.

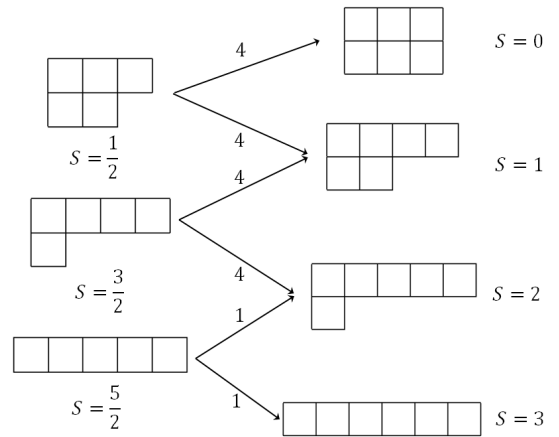


FIG. 1: Spin Young tableau of q^5 and q^5Q .

B. Flavor, color and spin state of q^5

After $SU(3)_F$ breaking, there are only two flavor bases for q^5 .

$$|F_1\rangle = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array} \right), |F_2\rangle = \left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 4 & 5 \\ \hline \end{array} \right). \quad (19)$$

We can construct the wave function of q^5 using the same method as in Sec. VB

- $F=[\bar{3}]$: Young tableau of color and spin state is $[3,2]$.

$$\begin{aligned} \psi_1 = & \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} - \frac{\sqrt{3}}{2} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \right)_{CS} \\ & - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 2 & 3 & 4 \\ \hline \end{array} - \frac{\sqrt{3}}{2} \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \right)_{CS}. \end{aligned} \quad (20)$$

- $F=[6]$: Young tableau of color and spin state is $[3,1,1]$.

$$\begin{aligned} \psi_2 = & \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} - \frac{\sqrt{5}}{\sqrt{8}} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline 5 & & \\ \hline \end{array} \right)_{CS} \\ & - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\frac{\sqrt{3}}{\sqrt{8}} \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} - \frac{\sqrt{5}}{\sqrt{8}} \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array} \right)_{CS}. \end{aligned} \quad (21)$$

- $F=[\bar{15}]$: Young tableau of color and spin state is $[2,2,1]$.

$$\begin{aligned} \psi_3 = & \frac{1}{\sqrt{2}} |F_1\rangle \otimes \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} - \frac{\sqrt{3}}{2} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} \right)_{CS} \\ & - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array} - \frac{\sqrt{3}}{2} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array} \right)_{CS}. \end{aligned} \quad (22)$$

- $F=[24]$: Young tableau of color and spin state is $[2,1,1,1]$.

$$\psi_4 = \frac{1}{\sqrt{2}} |F_1\rangle \otimes \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS} - \frac{1}{\sqrt{2}} |F_2\rangle \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS}. \quad (23)$$

VII. FLAVOR, COLOR AND SPIN STATE OF $q^5 Q$

In the appendix, we represent the color and spin state of q^5 and Clebsch-Gordan coefficients, from which we can construct the color and spin state of $q^5 Q$ using the following basis functions.

A. Flavor basis function

Since the isospin of the dibaryon is $\frac{1}{2}$, there are two flavor basis functions.

$$|F_1\rangle = \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right), |F_2\rangle = \left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \begin{array}{|c|} \hline 4 \\ \hline 5 \\ \hline \end{array}, \begin{array}{|c|} \hline 6 \\ \hline \end{array} \right). \quad (24)$$

B. Color basis function

Color singlet : five basis functions with Young tableau $[2,2,2]$

$$\begin{aligned} |C_1\rangle = & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} & |C_2\rangle = & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array} & |C_3\rangle = & \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array} & |C_4\rangle = & \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \\ |C_5\rangle = & \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array} \end{aligned} \quad (25)$$

C. Spin basis function

- $S=0$: five basis functions with Young tableau $[3,3]$

$$\begin{aligned} |S_1^0\rangle = & \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} & |S_2^0\rangle = & \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} & |S_3^0\rangle = & \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array} \\ |S_4^0\rangle = & \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} & |S_5^0\rangle = & \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array} \end{aligned}$$

- $S=1$: nine basis functions with Young tableau $[4,2]$

$$\begin{aligned} |S_1^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \hline \end{array} & |S_2^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & & \\ \hline \end{array} & |S_3^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & & \\ \hline \end{array} \\ |S_4^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & & \\ \hline \end{array} & |S_5^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline \end{array} & |S_6^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & & \\ \hline \end{array} \\ |S_7^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & & \\ \hline \end{array} & |S_8^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & & \\ \hline \end{array} & |S_9^1\rangle = & \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \end{aligned}$$

- $S=2$: five basis functions with Young tableau $[5,1]$

$$\begin{aligned} |S_1^2\rangle = & \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & & & & \\ \hline \end{array} & |S_2^2\rangle = & \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 6 \\ \hline 5 & & & & \\ \hline \end{array} \\ |S_3^2\rangle = & \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 \\ \hline 4 & & & & \\ \hline \end{array} & |S_4^2\rangle = & \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 6 \\ \hline 3 & & & & \\ \hline \end{array} \end{aligned}$$

$$|S_5^2\rangle = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & & & 6 \\ \hline \end{array}$$

- S=3 : one basis function with Young tableau [6]

$$|S^3\rangle = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$

D. Flavor, color and spin state of q^5Q

Here, we can construct the wave function of q^5Q from the state of q^5 . Since the color state of q^5 is $[3]$, there is only one way to construct color singlet state by adding one heavy flavor. There is no change in color basis function from q^5 to q^5Q . However, we should treat the spin state transformation carefully. The spin basis function of q^5 for $S=\frac{1}{2}$ can be either $S=0$ or $S=1$ of q^5Q state. We represent the spin basis transformation for each cases in the appendix.

- S=0 : 4 possible states

$$\begin{aligned} \psi_{1,S=\frac{1}{2}}^5 &\rightarrow \psi_{1,S=0}^6, \quad \psi_{2,S=\frac{1}{2}}^5 \rightarrow \psi_{2,S=0}^6, \\ \psi_{3,S=\frac{1}{2}}^5 &\rightarrow \psi_{3,S=0}^6, \quad \psi_{4,S=\frac{1}{2}}^5 \rightarrow \psi_{4,S=0}^6. \end{aligned}$$

- S=1 : 8 possible states

$$\begin{aligned} \psi_{1,S=\frac{1}{2}}^5 &\rightarrow \psi_{1,S=1}^6, \quad \psi_{2,S=\frac{1}{2}}^5 \rightarrow \psi_{2,S=1}^6, \\ \psi_{3,S=\frac{1}{2}}^5 &\rightarrow \psi_{3,S=1}^6, \quad \psi_{4,S=\frac{1}{2}}^5 \rightarrow \psi_{4,S=1}^6, \\ \psi_{1,S=\frac{3}{2}}^5 &\rightarrow \psi_{5,S=1}^6, \quad \psi_{2,S=\frac{3}{2}}^5 \rightarrow \psi_{6,S=1}^6, \\ \psi_{3,S=\frac{3}{2}}^5 &\rightarrow \psi_{7,S=1}^6, \quad \psi_{4,S=\frac{3}{2}}^5 \rightarrow \psi_{8,S=1}^6. \end{aligned}$$

- S=2 : 5 possible states

$$\begin{aligned} \psi_{1,S=\frac{3}{2}}^5 &\rightarrow \psi_{1,S=2}^6, \quad \psi_{2,S=\frac{3}{2}}^5 \rightarrow \psi_{2,S=2}^6, \\ \psi_{3,S=\frac{3}{2}}^5 &\rightarrow \psi_{3,S=2}^6, \quad \psi_{4,S=\frac{3}{2}}^5 \rightarrow \psi_{4,S=2}^6, \\ \psi_{3,S=\frac{5}{2}}^5 &\rightarrow \psi_{5,S=2}^6. \end{aligned}$$

- S=3 : 1 possible state

$$\psi_{3,S=\frac{5}{2}}^5 \rightarrow \psi_{1,S=3}^6.$$

We find that there is an orthogonal transformation between $\{\psi_i\}$ and $\{\phi_i\}$.

- S=0

$$\phi_1 = \psi_1, \quad \phi_2 = -\psi_2, \quad \phi_3 = \psi_3, \quad \phi_4 = \psi_4.$$

- S=1

$$\begin{aligned} \frac{2}{\sqrt{5}}\phi_1 - \frac{1}{\sqrt{5}}\phi_5 &= \psi_1, \quad \frac{1}{\sqrt{5}}\phi_1 + \frac{2}{\sqrt{5}}\phi_5 = \psi_5, \\ -\frac{\sqrt{5}}{3}\phi_2 + \frac{2}{3}\phi_6 &= \psi_2, \quad \frac{2}{3}\phi_2 + \frac{\sqrt{5}}{3}\phi_6 = \psi_6, \\ \frac{\sqrt{5}}{3}\phi_3 + \frac{2}{3}\phi_7 &= \psi_3, \quad \frac{2}{3}\phi_3 - \frac{\sqrt{5}}{3}\phi_7 = \psi_7, \\ -\frac{2}{3}\phi_4 - \frac{\sqrt{5}}{3}\phi_8 &= \psi_4, \quad -\frac{\sqrt{5}}{3}\phi_4 + \frac{2}{3}\phi_8 = \psi_8. \end{aligned}$$

- S=2

$$\begin{aligned} \phi_1 &= -\psi_1, \quad \phi_2 = \psi_2, \quad \phi_4 = -\psi_4, \\ -\frac{4}{5}\phi_3 + \frac{3}{5}\phi_5 &= \psi_3, \quad -\frac{3}{5}\phi_3 - \frac{4}{5}\phi_5 = \psi_5. \end{aligned}$$

- S=3

$$\phi_1 = \psi_1.$$

Since we calculate the expectation values using these wave functions, the results from both methods are the same as expected.

E. Baryon-baryon configuration of q^5Q

Considering the decay channel, we construct the baryon-baryon wave function of q^5Q . Among the five color basis functions, $|C_5\rangle$ is color singlet for 1,2,3 and at the same time color singlet for 4,5,6. Therefore, we have to construct the wave function only by using $|C_5\rangle$ as color state if we want to investigate decay channel through uud baryon and ssQ baryon. As done in Sec. VI B, we can construct the color, flavor and spin state of q^5Q which has the specific symmetry property for $S=0,1,2$.

$$\begin{aligned} \psi_{S=0}^{BB} &= \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(\frac{1}{2}|S_2^0\rangle + \frac{\sqrt{3}}{2}|S_4^0\rangle\right) \\ &\quad + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(\frac{1}{2}|S_3^0\rangle + \frac{\sqrt{3}}{2}|S_5^0\rangle\right). \end{aligned} \quad (26)$$

$$\begin{aligned} \psi_{S=1}^{BB} &= \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes (A_1|S_3^1\rangle + A_2(\frac{1}{2}|S_6^1\rangle + \frac{\sqrt{3}}{2}|S_8^1\rangle)) \\ &\quad + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes (A_1|S_4^1\rangle + A_2(\frac{1}{2}|S_7^1\rangle + \frac{\sqrt{3}}{2}|S_9^1\rangle)). \end{aligned} \quad (27)$$

$$\psi_{S=2}^{BB} = \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle. \quad (28)$$

In the above expressions, A_1 and A_2 are undetermined constants. However, to construct $S=1$ dibaryon, the spin of ssQ baryon can be either $\frac{1}{2}$ or $\frac{3}{2}$. When the spin of ssQ baryon is $\frac{3}{2}$, it should have the symmetry property $\{456\}$. Using this symmetry property, we can decide A_1 and A_2 . Once this state is determined, we can obtain the other state which is consist of ssQ baryon ($S = \frac{1}{2}$) by using the orthogonality of wave function.

- $uud(S = \frac{1}{2}) + ssQ(S = \frac{1}{2})$

$$\begin{aligned} \psi_{S=1}^{BB} &= \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes \left(-\frac{2\sqrt{2}}{3}|S_3^1\rangle + \frac{1}{6}|S_6^1\rangle + \frac{\sqrt{3}}{6}|S_8^1\rangle\right) \\ &\quad + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes \left(-\frac{2\sqrt{2}}{3}|S_4^1\rangle + \frac{1}{6}|S_7^1\rangle + \frac{\sqrt{3}}{6}|S_9^1\rangle\right). \end{aligned} \quad (29)$$

- $uud(S = \frac{1}{2}) + ssQ(S = \frac{3}{2})$

$$\begin{aligned} \psi_{S=1}^{BB} = & \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes (\frac{1}{3}|S_3^1\rangle + \frac{\sqrt{2}}{3}|S_6^1\rangle + \frac{\sqrt{6}}{3}|S_8^1\rangle) \\ & + \frac{1}{\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes (\frac{1}{3}|S_4^1\rangle + \frac{\sqrt{2}}{3}|S_7^1\rangle + \frac{\sqrt{6}}{3}|S_9^1\rangle). \end{aligned} \quad (30)$$

We can not construct the baryon{123}-baryon{456} wave function with $S=3$, because there is no uud baryon with $I = \frac{1}{2}$ and $S = \frac{3}{2}$.

VIII. NUMERICAL RESULTS

In this section, we present our numerical results. Before using variational method, we can estimate the stability condition only by using the simple color-spin interaction formula given as $-\sum_{i<j}^N 1/(m_i m_j) \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$ without taking into account the r-dependence of the hyperfine potential. In the upper part of Table IV, we show the matrix elements without the masses, which reflects the attraction in the SU(4) flavor symmetric limit. As one can see here, the $\psi_{5,S=1}$ is the most attractive channel. The lower part of Table IV is the realistic case where all the relevant constituent quark masses are included. One notes that for $S=0,1$, $\Lambda\Xi_c$ system is now more attractive than anti-triplet dibaryon state, but for $S=2$, the dibaryon state is more attractive. Therefore, we can expect to have a compact stable dibaryon state for $S=2$ configuration when considering the color-spin interaction only.

We now analyse the numerical results obtained from the variational method. Varying the parameters of spatial function, we find the ground state of the dibaryon. The parameters and masses are in Table V. Here, we represent the color, flavor and spin wave function of ground state dibaryon obtained from variational method for $S=0,1,2,3$.

$$\begin{aligned} \psi_{S=0}^{\text{ground}} &= -0.5\psi_1^6 + 0.71\psi_2^6 - 0.17\psi_3^6 - 0.47\psi_4^6 \\ &= 0.35|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + 0.35|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle \\ &+ 0.61|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + 0.61|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle. \end{aligned} \quad (31)$$

$$\begin{aligned} \psi_{S=1}^{\text{ground}} &= -0.17\psi_1^6 + 0.24\psi_2^6 - 0.06\psi_3^6 - 0.16\psi_4^6 \\ &+ 0.67\psi_5^6 + 0.42\psi_6^6 - 0.5\psi_7^6 + 0.14\psi_8^6 \\ &= -0.67|C_5\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle - 0.67|C_5\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle \\ &+ 0.12|C_5\rangle \otimes |F_1\rangle \otimes |S_6^1\rangle + 0.12|C_5\rangle \otimes |F_2\rangle \otimes |S_7^1\rangle \\ &+ 0.2|C_5\rangle \otimes |F_1\rangle \otimes |S_8^1\rangle + 0.2|C_5\rangle \otimes |F_2\rangle \otimes |S_9^1\rangle. \end{aligned} \quad (32)$$

TABLE IV: The expectation values of $-\sum_{i<j}^N \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$ and $-\sum_{i<j}^N 1/(m_i m_j) \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$ for each anti-triplet flavor states and corresponding decay channel

$-\sum_{i<j}^N \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$				
$uudssc$	$\psi_{1,S=0}$	$\psi_{1,S=1}$	$\psi_{5,S=1}$	$\psi_{1,S=2}$
	-24	$-\frac{40}{3}$	$-\frac{76}{3}$	-4
decay channel	$N\Omega_c$	$N\Omega_c$	$N\Omega_c$	$N\Omega_c^*$
	-16	-16	-16	0
decay channel	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi_c^*$
	-16	-16	-16	0

$-\sum_{i<j}^N 1/(m_i m_j) \langle \lambda_i^c \lambda_j^c \sigma_i \cdot \sigma_j \rangle$				
$uudssc$	$\psi_{1,S=0}$	$\psi_{1,S=1}$	$\psi_{5,S=1}$	$\psi_{1,S=2}$
	-98.1	-87.1	-91.2	-64.9
decay channel	$N\Omega_c$	$N\Omega_c$	$N\Omega_c$	$N\Omega_c^*$
	-70.1	-70.1	-70.1	-57
decay channel	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi_c$	$\Lambda\Xi_c^*$
	-104.9	-104.9	-104.9	-49.5

$$\begin{aligned} \psi_{S=2}^{\text{ground}} &= -0.71\psi_1^6 - 0.45\psi_2^6 + 0.53\psi_3^6 - 0.15\psi_4^6 \\ &= 0.71|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + 0.71|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle. \end{aligned} \quad (33)$$

$$\begin{aligned} \psi_{S=3}^{\text{ground}} &= \psi_1^6 \\ &= \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^3\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^3\rangle \\ &- \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^3\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^3\rangle. \end{aligned} \quad (34)$$

As we can see, the color wave function of the ground state for $S=0,1,2$ is $|C_5\rangle$ and the coefficients are almost the same as in Sec. VII E. Therefore, the ground state of q^5Q for $S=0,1,2$ is the sum of two baryon states. However, for $S=3$, the color state is not $|C_5\rangle$ because there is no baryon state with $I=1/2$ and $S=3/2$ for uud . Therefore, the ground state for $S=3$ is not the sum of two baryon states. Also, as can be seen in Table V, $d = 0$ for the ground state other than the $S=3$. This parameter corresponds to a well separated baryon baryon configuration. As $uudssQ$ has the $\{123\}\{45\}6$ symmetry property, even with a single Gaussian trial wave function given in Eq. 11, one can express the wave function for a well separated baryon(123)-baryon(456) configuration.

Additionally, we have to notice that there are two cases to make $S=1$ dibaryon from the two baryon states. The first one is two $S=1/2$ baryons and second is one $S=1/2$

TABLE V: Variational parameters fitted to the ground state of the dibaryon using the variational method. The other parameters are the same as baryon. The mass unit is GeV.

uudssc	a	b	c	d	Mass	Decay mode
S=0	2.5	3.5	5.7	0	3.666	$N\Omega_c$
S=1	2.5	3.5	5.7	0	3.666	$N\Omega_c$
S=2	2.5	3.2	4.8	0	3.737	$N\Omega_c^*$
S=3	1.6	3.3	2.8	2.4	4.404	$\Sigma^*\Xi_c^*$

uudssb	a	b	c	d	Mass	Decay mode
S=0	2.5	3.5	6.5	0	6.997	$N\Omega_b$
S=1	2.5	3.5	6.5	0	6.997	$N\Omega_b$
S=2	2.5	3.3	6.1	0	7.028	$N\Omega_b^*$
S=3	1.6	3.3	4.5	2.5	7.803	$\Sigma^*\Xi_b^*$

baryon and one S=3/2 baryon. The ground state in Eq. (32) corresponds to the first case. We can also calculate the excited state for S=1 by using variational parameters obtained from the ground state. We find that the flavor color spin wave function for the first excited state is given as

$$\begin{aligned}
\psi_{S=1}^{\text{excited}} &= 0.24|C_5\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle + 0.24|C_5\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle \\
&+ 0.33|C_5\rangle \otimes |F_1\rangle \otimes |S_6^1\rangle + 0.33|C_5\rangle \otimes |F_2\rangle \otimes |S_7^1\rangle \\
&+ 0.58|C_5\rangle \otimes |F_1\rangle \otimes |S_8^1\rangle + 0.58|C_5\rangle \otimes |F_2\rangle \otimes |S_9^1\rangle.
\end{aligned} \tag{35}$$

This form turns out to be the same as the second case for S=1 given in Sec.VIIE, which represents flavor color spin state of the $N\Omega_c^*$ system.

IX. SUMMARY

In this work, we investigate the symmetry property and the stability of the dibaryon containing two strange quarks and one heavy flavor with $I = \frac{1}{2}$. To obtain the wave function of the dibaryon with required symmetry property, we utilize two methods; the first one is to construct the color and spin wave function of the dibaryon from each flavor state directly, and the second one is to form the color and spin state of q^5 first, and then construct the wave function of the dibaryon by adding one heavy quark. We verify that their results are the same.

In the SU(3) flavor limit, by using Eq. (8), we expect the flavor anti-triplet state to be the most attractive channel when five light quarks are considered. Additionally, if we increase the mass of sixth quark, then its additional kinetic term will become smaller so that it may lead more chance to form the stable and compact dibaryon state. Hence, we calculate the mass of the

dibaryon with two strange quarks and one heavy flavor.

We first estimate the stability condition by using only the color-spin interaction without the r-dependence. In the SU(4) flavor symmetric limit, all the anti-triplet flavor states of the dibaryon with S=0,1,2 are more attractive than the decay channel. However, when we consider the relevant constituent quark masses obtained from baryon fitting results, only the anti-triplet state for S=2 is more attractive.

Finally, we calculate the masses of the dibaryons in a nonrelativistic Hamiltonian by using variational method. We find that the ground state of the dibaryon for S=0,1,2 is the well separated two baryon states. As for S=3, with our single Gaussian form, the mass of the ground state is found to be more repulsive than the decay channel. Hence, we conclude that there are no stable and compact $uudssQ$ dibaryon state with $I = \frac{1}{2}$ within the given potential. If we consider the dibaryon containing more than one heavy quark, then their additional kinetic terms will decrease further than the dibaryon with one heavy quark so that it may lead more chance to form the stable dibaryon.

Acknowledgements

This work was supported by the Korea National Research Foundation under the grant number KRF-2011-0020333 and KRF-2011-0030621.

Appendix A: Color and Spin state of q^5

Here, we represent the color and spin basis function of q^5 .

1. Color basis function

$$\begin{aligned}
|C_1\rangle &= \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array} & |C_2\rangle &= \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array} & |C_3\rangle &= \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array} & |C_4\rangle &= \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} \\
|C_5\rangle &= \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} & & & & &
\end{aligned} \tag{A1}$$

2. Spin basis function

- S=1/2 : 5 basis functions with Young tableau [3,3]

$$|S_1^{\frac{1}{2}}\rangle = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \quad |S_2^{\frac{1}{2}}\rangle = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \quad |S_3^{\frac{1}{2}}\rangle = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array}$$

$$|S_4^{\frac{1}{2}}\rangle = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \quad |S_5^{\frac{1}{2}}\rangle = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}$$

• S=5/2 : 1 basis function with Young tableau [6]

$$|S_1^{\frac{5}{2}}\rangle = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array}$$

• S=3/2 : 4 basis functions with Young tableau [4,2]

$$|S_1^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & & & \\ \hline \end{array} \quad |S_2^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & & & \\ \hline \end{array} \quad |S_3^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & & & \\ \hline \end{array}$$

$$|S_4^{\frac{3}{2}}\rangle = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & & & \\ \hline \end{array}$$

Appendix B: CS coupling of $q^5 Q$

In this section, we present the color \otimes spin basis. The Clebsch-Gordon coefficients of combining the color and spin basis are calculated by using K matrix [15, 16].

1. S=0

$$\begin{aligned} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_1\rangle \otimes |S_2^0\rangle + \frac{1}{2}|C_2\rangle \otimes |S_3^0\rangle + \frac{1}{2}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2}|C_4\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_1\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^0\rangle - \frac{1}{2}|C_5\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_2\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^0\rangle + \frac{1}{2}|C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_3\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_3^0\rangle + \frac{1}{2}|C_5\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_4\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_2^0\rangle - \frac{1}{2}|C_5\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_5^0\rangle \end{aligned}$$

$$\begin{aligned} \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & & & \\ \hline 6 & & & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{2}}|C_1\rangle \otimes |S_4^0\rangle + \frac{1}{\sqrt{2}}|C_2\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & & & \\ \hline 6 & & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_2^0\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_3^0\rangle + \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_4^0\rangle \\ &\quad + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & & & \\ \hline 6 & & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_1^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_3^0\rangle \\ &\quad - \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_5^0\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_5^0\rangle \\ \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & & & \\ \hline 6 & & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_1^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_2^0\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_3^0\rangle \\ &\quad + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_4^0\rangle + \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_5^0\rangle \end{aligned}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline \end{array} \quad CS = -\frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle \otimes |S_2^0\rangle - \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_2^0\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle \otimes |S_3^0\rangle - \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle \otimes |S_4^0\rangle \\ + \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & & & \\ \hline 5 & & & \\ \hline \end{array} \quad CS = -\frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle \otimes |S_1^0\rangle + \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_1^0\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_3^0\rangle - \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_3^0\rangle \\ - \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_5^0\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & & & \\ \hline 5 & & & \\ \hline \end{array} \quad CS = -\frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle \otimes |S_1^0\rangle + \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_1^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_2^0\rangle + \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_3^0\rangle \\ + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & & & \\ \hline 4 & & & \\ \hline \end{array} \quad CS = \frac{1}{2}|C_1\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^0\rangle \\ + \frac{1}{2}|C_5\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & & & \\ \hline 4 & & & \\ \hline \end{array} \quad CS = \frac{1}{2}|C_2\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^0\rangle - \frac{1}{2}|C_5\rangle \otimes |S_4^0\rangle \\ + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 4 & 5 & 6 \\ \hline 2 & & & \\ \hline 3 & & & \\ \hline \end{array} \quad CS = -\frac{1}{2}|C_2\rangle \otimes |S_2^0\rangle + \frac{1}{3}|C_1\rangle \otimes |S_3^0\rangle - \frac{1}{2}|C_4\rangle \otimes |S_4^0\rangle + \frac{1}{2}|C_3\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \quad CS = -\frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_3^0\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_4^0\rangle \\ + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \quad CS = -\frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_2^0\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_4^0\rangle \\ + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_5^0\rangle$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} \quad CS = -\frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_1^0\rangle - \frac{1}{3}|C_3\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_2^0\rangle - \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_3^0\rangle$$

$$\begin{aligned}
& + \frac{\sqrt{2}}{3} |C_4\rangle \otimes |S_3^0\rangle - \frac{1}{3} |C_5\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{6}} |C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{6}} |C_4\rangle \otimes |S_5^0\rangle + \frac{1}{2\sqrt{3}} |C_5\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} & CS \\
& = -\frac{1}{2\sqrt{3}} |C_2\rangle \otimes |S_1^0\rangle - \frac{1}{3} |C_4\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{6}} |C_2\rangle \otimes |S_2^0\rangle + \frac{\sqrt{2}}{3} |C_4\rangle \otimes |S_2^0\rangle + \frac{1}{3} |C_5\rangle \otimes |S_2^0\rangle \\
& - \frac{1}{2\sqrt{6}} |C_1\rangle \otimes |S_3^0\rangle + \frac{\sqrt{2}}{3} |C_3\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{6}} |C_4\rangle \otimes |S_4^0\rangle - \frac{1}{2\sqrt{3}} |C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{6}} |C_3\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline 6 & \\ \hline \end{array} & CS \\
& = \frac{2}{3} |C_5\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{3}} |C_2\rangle \otimes |S_2^0\rangle + \frac{1}{3} |C_4\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{3}} |C_1\rangle \otimes |S_3^0\rangle - \frac{1}{3} |C_3\rangle \otimes |S_3^0\rangle \\
& - \frac{1}{2\sqrt{3}} |C_4\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{3}} |C_3\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} & CS \\
& = -\frac{1}{\sqrt{6}} |C_1\rangle \otimes |S_1^0\rangle + \frac{1}{3\sqrt{2}} |C_3\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{3}} |C_1\rangle \otimes |S_2^0\rangle + \frac{1}{3} |C_3\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{3}} |C_2\rangle \otimes |S_3^0\rangle \\
& - \frac{1}{3} |C_4\rangle \otimes |S_3^0\rangle + \frac{1}{3\sqrt{2}} |C_5\rangle \otimes |S_3^0\rangle - \frac{1}{2\sqrt{3}} |C_3\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{3}} |C_4\rangle \otimes |S_5^0\rangle + \frac{1}{\sqrt{6}} |C_5\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} & CS \\
& = -\frac{1}{\sqrt{6}} |C_2\rangle \otimes |S_1^0\rangle + \frac{1}{3\sqrt{2}} |C_4\rangle \otimes |S_1^0\rangle - \frac{1}{2\sqrt{3}} |C_2\rangle \otimes |S_2^0\rangle - \frac{1}{3} |C_4\rangle \otimes |S_2^0\rangle - \frac{1}{3\sqrt{2}} |C_5\rangle \otimes |S_2^0\rangle \\
& - \frac{1}{2\sqrt{3}} |C_1\rangle \otimes |S_3^0\rangle - \frac{1}{3} |C_3\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{3}} |C_4\rangle \otimes |S_4^0\rangle - \frac{1}{\sqrt{6}} |C_5\rangle \otimes |S_4^0\rangle + \frac{1}{2\sqrt{3}} |C_3\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array} & CS \\
& = -\frac{\sqrt{2}}{3} |C_5\rangle \otimes |S_1^0\rangle + \frac{1}{\sqrt{6}} |C_2\rangle \otimes |S_2^0\rangle - \frac{1}{3\sqrt{2}} |C_4\rangle \otimes |S_2^0\rangle - \frac{1}{\sqrt{6}} |C_1\rangle \otimes |S_3^0\rangle + \frac{1}{3\sqrt{2}} |C_3\rangle \otimes |S_3^0\rangle \\
& - \frac{1}{\sqrt{6}} |C_4\rangle \otimes |S_4^0\rangle + \frac{1}{\sqrt{6}} |C_3\rangle \otimes |S_5^0\rangle \\
\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} & CS \\
& = \frac{\sqrt{2}}{\sqrt{15}} |C_5\rangle \otimes |S_1^0\rangle - \frac{\sqrt{2}}{\sqrt{15}} |C_4\rangle \otimes |S_2^0\rangle + \frac{\sqrt{2}}{\sqrt{15}} |C_3\rangle \otimes |S_3^0\rangle - \frac{\sqrt{3}}{\sqrt{10}} |C_2\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{\sqrt{10}} |C_1\rangle \otimes |S_5^0\rangle
\end{aligned}$$

2. S=1

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array} & CS \\
& = -\frac{1}{2\sqrt{10}} |C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{3}}{2\sqrt{10}} |C_3\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{10}} |C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{3}}{2\sqrt{10}} |C_4\rangle \otimes |S_4^1\rangle + \frac{1}{\sqrt{5}} |C_1\rangle \otimes |S_6^1\rangle \\
& + \frac{1}{\sqrt{5}} |C_2\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{5}} |C_3\rangle \otimes |S_8^1\rangle + \frac{1}{\sqrt{5}} |C_4\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline 6 & & \\ \hline \end{array} \quad CS
= -\frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_2^1\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{4\sqrt{5}}|C_1\rangle \otimes |S_3^1\rangle + \frac{1}{4\sqrt{5}}|C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{5}}|C_1\rangle \otimes |S_5^1\rangle + \frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_7^1\rangle - \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_8^1\rangle + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_9^1\rangle$$

$$-\frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & & \\ \hline \end{array} \quad CS
= -\frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_2^1\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{4\sqrt{5}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{4\sqrt{5}}|C_1\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{5}}|C_2\rangle \otimes |S_5^1\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_8^1\rangle + \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_8^1\rangle$$

$$+\frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array} \quad CS
= \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_1^1\rangle + \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{4\sqrt{5}}|C_3\rangle \otimes |S_3^1\rangle + \frac{1}{4\sqrt{5}}|C_4\rangle \otimes |S_4^1\rangle + \frac{1}{2\sqrt{10}}|C_5\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_5^1\rangle - \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_6^1\rangle + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_7^1\rangle - \frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_8^1\rangle$$

$$+\frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline 6 & & \\ \hline \end{array} \quad CS
= \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_1^1\rangle + \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{4\sqrt{5}}|C_4\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{10}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{4\sqrt{5}}|C_3\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_5^1\rangle + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_6^1\rangle + \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_8^1\rangle$$

$$+\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 6 & \\ \hline 5 & & \\ \hline \end{array} \quad CS
= \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_4^1\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |S_6^1\rangle$$

$$+\frac{2}{3\sqrt{5}}|C_3\rangle \otimes |S_6^1\rangle + \frac{1}{\sqrt{15}}|C_2\rangle \otimes |S_6^1\rangle + \frac{2}{3\sqrt{5}}|C_4\rangle \otimes |S_7^1\rangle - \frac{1}{\sqrt{15}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{15}}|C_4\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & & \\ \hline \end{array} \quad CS
= \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{15}}|C_1\rangle \otimes |S_5^1\rangle - \frac{2}{3\sqrt{5}}|C_3\rangle \otimes |S_5^1\rangle + \frac{1}{\sqrt{30}}|C_1\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{30}}|C_2\rangle \otimes |S_7^1\rangle + \frac{2}{3\sqrt{5}}|C_5\rangle \otimes |S_7^1\rangle$$

$$+\frac{1}{\sqrt{30}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{30}}|C_4\rangle \otimes |S_9^1\rangle + \frac{1}{\sqrt{15}}|C_5\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & \\ \hline 5 & & \\ \hline \end{array} \quad CS
= \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle$$

$$+\frac{1}{\sqrt{15}}|C_2\rangle \otimes |S_5^1\rangle - \frac{2}{3\sqrt{5}}|C_4\rangle \otimes |S_5^1\rangle - \frac{1}{\sqrt{30}}|C_2\rangle \otimes |S_6^1\rangle - \frac{2}{3\sqrt{5}}|C_5\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{30}}|C_1\rangle \otimes |S_7^1\rangle$$

$$-\frac{1}{\sqrt{30}}|C_4\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{15}}|C_5\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{30}}|C_3\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline \end{array} \quad CS$$

$$= \frac{\sqrt{5}}{3\sqrt{2}}|C_3\rangle \otimes |S_1^1\rangle + \frac{1}{6\sqrt{2}}|C_1\rangle \otimes |S_2^1\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{12}|C_1\rangle \otimes |S_3^1\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_3^1\rangle$$

$$+ \frac{1}{12}|C_2\rangle \otimes |S_4^1\rangle - \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_4^1\rangle + \frac{1}{2\sqrt{6}}|C_5\rangle \otimes |S_4^1\rangle - \frac{1}{3}|C_1\rangle \otimes |S_5^1\rangle + \frac{1}{3\sqrt{2}}|C_1\rangle \otimes |S_6^1\rangle$$

$$- \frac{1}{3\sqrt{2}}|C_2\rangle \otimes |S_7^1\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_9^1\rangle - \frac{1}{3}|C_5\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & & \\ \hline \end{array} \quad CS$$

$$= \frac{\sqrt{5}}{3\sqrt{2}}|C_4\rangle \otimes |S_1^1\rangle + \frac{1}{6\sqrt{2}}|C_2\rangle \otimes |S_2^1\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{12}|C_2\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_3^1\rangle$$

$$- \frac{1}{2\sqrt{6}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{12}|C_1\rangle \otimes |S_4^1\rangle - \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_4^1\rangle - \frac{1}{3}|C_2\rangle \otimes |S_5^1\rangle - \frac{1}{3\sqrt{2}}|C_2\rangle \otimes |S_6^1\rangle$$

$$- \frac{1}{3\sqrt{2}}|C_1\rangle \otimes |S_7^1\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_8^1\rangle + \frac{1}{3}|C_5\rangle \otimes |S_8^1\rangle - \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 & \\ \hline 3 & & \\ \hline \end{array} \quad CS$$

$$= \frac{\sqrt{5}}{3\sqrt{2}}|C_5\rangle \otimes |S_1^1\rangle - \frac{1}{\sqrt{6}}|C_5\rangle \otimes |S_2^1\rangle - \frac{1}{6\sqrt{2}}|C_2\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_3^1\rangle + \frac{1}{6\sqrt{2}}|C_1\rangle \otimes |S_4^1\rangle$$

$$+ \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_4^1\rangle + \frac{1}{3}|C_2\rangle \otimes |S_6^1\rangle - \frac{1}{3}|C_1\rangle \otimes |S_7^1\rangle + \frac{1}{3}|C_4\rangle \otimes |S_8^1\rangle - \frac{1}{3}|C_3\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & \\ \hline 5 & & \\ \hline \end{array} \quad CS$$

$$= \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_3\rangle \otimes |S_2^1\rangle + \frac{5}{12\sqrt{3}}|C_3\rangle \otimes |S_3^1\rangle - \frac{5}{12\sqrt{3}}|C_4\rangle \otimes |S_4^1\rangle - \frac{5}{6\sqrt{6}}|C_5\rangle \otimes |S_4^1\rangle$$

$$- \frac{1}{3\sqrt{3}}|C_3\rangle \otimes |S_5^1\rangle + \frac{1}{3\sqrt{6}}|C_3\rangle \otimes |S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_4\rangle \otimes |S_7^1\rangle - \frac{1}{3\sqrt{3}}|C_5\rangle \otimes |S_7^1\rangle - \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_8^1\rangle$$

$$+ \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \hline \end{array} \quad CS$$

$$= \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_4\rangle \otimes |S_2^1\rangle - \frac{5}{12\sqrt{3}}|C_4\rangle \otimes |S_3^1\rangle + \frac{5}{6\sqrt{6}}|C_5\rangle \otimes |S_3^1\rangle - \frac{5}{12\sqrt{3}}|C_3\rangle \otimes |S_4^1\rangle$$

$$- \frac{1}{3\sqrt{3}}|C_4\rangle \otimes |S_5^1\rangle - \frac{1}{3\sqrt{6}}|C_4\rangle \otimes |S_6^1\rangle + \frac{1}{3\sqrt{3}}|C_5\rangle \otimes |S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_3\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_8^1\rangle$$

$$+ \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad CS$$

$$= -\frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle \otimes |S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_1\rangle \otimes |S_2^1\rangle + \frac{5}{18\sqrt{2}}|C_3\rangle \otimes |S_2^1\rangle + \frac{5}{12\sqrt{3}}|C_1\rangle \otimes |S_3^1\rangle + \frac{5}{18}|C_3\rangle \otimes |S_3^1\rangle$$

$$- \frac{5}{12\sqrt{3}}|C_2\rangle \otimes |S_4^1\rangle - \frac{5}{18}|C_4\rangle \otimes |S_4^1\rangle + \frac{5}{18\sqrt{2}}|C_5\rangle \otimes |S_4^1\rangle - \frac{1}{3\sqrt{3}}|C_1\rangle \otimes |S_5^1\rangle - \frac{2}{9}|C_3\rangle \otimes |S_5^1\rangle$$

$$+ \frac{1}{3\sqrt{6}}|C_1\rangle \otimes |S_6^1\rangle - \frac{2\sqrt{2}}{9}|C_3\rangle \otimes |S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_2\rangle \otimes |S_7^1\rangle + \frac{2\sqrt{2}}{9}|C_4\rangle \otimes |S_7^1\rangle - \frac{2}{9}|C_5\rangle \otimes |S_7^1\rangle$$

$$- \frac{1}{3\sqrt{6}}|C_3\rangle \otimes |S_8^1\rangle + \frac{1}{3\sqrt{6}}|C_4\rangle \otimes |S_9^1\rangle + \frac{1}{3\sqrt{3}}|C_5\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} \quad CS$$

$$= -\frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle \otimes |S_1^1\rangle - \frac{5}{6\sqrt{6}}|C_2\rangle \otimes |S_2^1\rangle + \frac{5}{18\sqrt{2}}|C_4\rangle \otimes |S_2^1\rangle - \frac{5}{12\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle - \frac{5}{18}|C_4\rangle \otimes |S_3^1\rangle$$

$$- \frac{5}{18\sqrt{2}}|C_5\rangle \otimes |S_3^1\rangle - \frac{5}{12\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle - \frac{5}{18}|C_3\rangle \otimes |S_4^1\rangle - \frac{1}{3\sqrt{3}}|C_2\rangle \otimes |S_5^1\rangle - \frac{2}{9}|C_4\rangle \otimes |S_5^1\rangle$$

$$\begin{aligned}
& -\frac{1}{3\sqrt{6}}|C_2\rangle \otimes |S_6^1\rangle + \frac{2\sqrt{2}}{9}|C_4\rangle \otimes |S_6^1\rangle + \frac{2}{9}|C_5\rangle \otimes |S_6^1\rangle - \frac{1}{3\sqrt{6}}|C_1\rangle \otimes |S_7^1\rangle + \frac{2\sqrt{2}}{9}|C_3\rangle \otimes |S_7^1\rangle \\
& + \frac{1}{3\sqrt{6}}|C_4\rangle \otimes |S_8^1\rangle - \frac{1}{3\sqrt{3}}|C_5\rangle \otimes |S_8^1\rangle + \frac{1}{3\sqrt{6}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array} &_{CS} = -\frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle \otimes |S_1^1\rangle - \frac{5}{9\sqrt{2}}|C_5\rangle \otimes |S_2^1\rangle + \frac{5}{6\sqrt{6}}|C_2\rangle \otimes |S_3^1\rangle - \frac{5}{18\sqrt{2}}|C_4\rangle \otimes |S_3^1\rangle - \frac{5}{6\sqrt{6}}|C_1\rangle \otimes |S_4^1\rangle \\
& + \frac{5}{18\sqrt{2}}|C_3\rangle \otimes |S_4^1\rangle + \frac{4}{9}|C_5\rangle \otimes |S_5^1\rangle + \frac{1}{3\sqrt{3}}|C_2\rangle \otimes |S_6^1\rangle + \frac{2}{9}|C_4\rangle \otimes |S_6^1\rangle - \frac{1}{3\sqrt{3}}|C_1\rangle \otimes |S_7^1\rangle \\
& - \frac{2}{9}|C_3\rangle \otimes |S_7^1\rangle - \frac{1}{3\sqrt{3}}|C_4\rangle \otimes |S_8^1\rangle + \frac{1}{3\sqrt{3}}|C_3\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} &_{CS} = \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_3^1\rangle - \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_3^1\rangle + \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_4^1\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_4^1\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_6^1\rangle \\
& - \frac{1}{3}|C_3\rangle \otimes |S_6^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_7^1\rangle - \frac{1}{3}|C_4\rangle \otimes |S_7^1\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} &_{CS} = \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_2^1\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_2^1\rangle + \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_4^1\rangle - \frac{1}{3\sqrt{2}}|C_5\rangle \otimes |S_4^1\rangle \\
& - \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_5^1\rangle + \frac{1}{3}|C_3\rangle \otimes |S_5^1\rangle - \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_7^1\rangle - \frac{1}{3}|C_5\rangle \otimes |S_7^1\rangle \\
& - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_9^1\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} &_{CS} = \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_2^1\rangle + \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_2^1\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle + \frac{1}{3\sqrt{2}}|C_5\rangle \otimes |S_3^1\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle \\
& - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_5^1\rangle + \frac{1}{3}|C_4\rangle \otimes |S_5^1\rangle + \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_6^1\rangle + \frac{1}{3}|C_5\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_7^1\rangle \\
& + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline 4 & & \\ \hline 6 & & \\ \hline \end{array} &_{CS} = \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_1^1\rangle + \frac{1}{3\sqrt{10}}|C_1\rangle \otimes |S_2^1\rangle + \frac{1}{\sqrt{30}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{6\sqrt{5}}|C_1\rangle \otimes |S_3^1\rangle - \frac{1}{\sqrt{15}}|C_3\rangle \otimes |S_3^1\rangle \\
& + \frac{1}{6\sqrt{5}}|C_2\rangle \otimes |S_4^1\rangle - \frac{1}{\sqrt{15}}|C_4\rangle \otimes |S_4^1\rangle + \frac{1}{\sqrt{30}}|C_5\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{6}|C_1\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_1\rangle \otimes |S_6^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{2}}|C_2\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_9^1\rangle + \frac{\sqrt{5}}{6}|C_5\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & & \\ \hline 4 & & \\ \hline 6 & & \\ \hline \end{array} &_{CS} = \frac{\sqrt{2}}{3}|C_4\rangle \otimes |S_1^1\rangle + \frac{1}{3\sqrt{10}}|C_2\rangle \otimes |S_2^1\rangle + \frac{1}{\sqrt{30}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{6\sqrt{5}}|C_2\rangle \otimes |S_3^1\rangle - \frac{1}{\sqrt{15}}|C_4\rangle \otimes |S_3^1\rangle \\
& - \frac{1}{\sqrt{30}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{6\sqrt{5}}|C_1\rangle \otimes |S_4^1\rangle - \frac{1}{\sqrt{15}}|C_3\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{6}|C_2\rangle \otimes |S_5^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_2\rangle \otimes |S_6^1\rangle
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sqrt{5}}{6\sqrt{2}}|C_1\rangle \otimes |S_7^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_8^1\rangle - \frac{\sqrt{5}}{6}|C_5\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & & \\ \hline 3 & & \\ \hline 6 & & \\ \hline \end{array} & = \frac{\sqrt{2}}{3}|C_5\rangle \otimes |S_1^1\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle \otimes |S_2^1\rangle - \frac{1}{3\sqrt{10}}|C_2\rangle \otimes |S_3^1\rangle - \frac{1}{\sqrt{30}}|C_4\rangle \otimes |S_3^1\rangle + \frac{1}{3\sqrt{10}}|C_1\rangle \otimes |S_4^1\rangle \\
CS & + \frac{1}{\sqrt{30}}|C_3\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{6}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{6}|C_1\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{6}|C_4\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & & \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} & = -\frac{2}{3\sqrt{3}}|C_3\rangle \otimes |S_1^1\rangle + \frac{4}{3\sqrt{15}}|C_1\rangle \otimes |S_2^1\rangle + \frac{2}{9\sqrt{5}}|C_3\rangle \otimes |S_2^1\rangle - \frac{2\sqrt{2}}{3\sqrt{15}}|C_1\rangle \otimes |S_3^1\rangle + \frac{2\sqrt{2}}{9\sqrt{5}}|C_3\rangle \otimes |S_3^1\rangle \\
CS & + \frac{2\sqrt{2}}{3\sqrt{15}}|C_2\rangle \otimes |S_4^1\rangle - \frac{2\sqrt{2}}{9\sqrt{5}}|C_4\rangle \otimes |S_4^1\rangle + \frac{2}{9\sqrt{5}}|C_5\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle \otimes |S_5^1\rangle \\
& - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_7^1\rangle + \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle \otimes |S_7^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_8^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_9^1\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & & \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} & = -\frac{2}{3\sqrt{3}}|C_4\rangle \otimes |S_1^1\rangle + \frac{4}{3\sqrt{15}}|C_2\rangle \otimes |S_2^1\rangle + \frac{2}{9\sqrt{5}}|C_4\rangle \otimes |S_2^1\rangle + \frac{2\sqrt{2}}{3\sqrt{15}}|C_2\rangle \otimes |S_3^1\rangle - \frac{2\sqrt{2}}{9\sqrt{5}}|C_4\rangle \otimes |S_3^1\rangle \\
CS & - \frac{2}{9\sqrt{5}}|C_5\rangle \otimes |S_3^1\rangle + \frac{2\sqrt{2}}{3\sqrt{15}}|C_1\rangle \otimes |S_4^1\rangle - \frac{2\sqrt{2}}{9\sqrt{5}}|C_3\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle \otimes |S_5^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_7^1\rangle + \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_7^1\rangle \\
& - \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle \otimes |S_8^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & & \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array} & = -\frac{2}{3\sqrt{3}}|C_5\rangle \otimes |S_1^1\rangle - \frac{4}{9\sqrt{5}}|C_5\rangle \otimes |S_2^1\rangle - \frac{4}{3\sqrt{15}}|C_2\rangle \otimes |S_3^1\rangle - \frac{2}{9\sqrt{5}}|C_4\rangle \otimes |S_3^1\rangle + \frac{4}{3\sqrt{15}}|C_1\rangle \otimes |S_4^1\rangle \\
CS & + \frac{2}{9\sqrt{5}}|C_3\rangle \otimes |S_4^1\rangle + \frac{\sqrt{10}}{9}|C_5\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle \otimes |S_7^1\rangle \\
& - \frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle \otimes |S_7^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle \otimes |S_8^1\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 5 & 6 \\ \hline 2 & & \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} & = \frac{2}{3\sqrt{3}}|C_5\rangle \otimes |S_2^1\rangle - \frac{2}{3\sqrt{3}}|C_4\rangle \otimes |S_3^1\rangle + \frac{2}{3\sqrt{3}}|C_3\rangle \otimes |S_4^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle \otimes |S_5^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle \otimes |S_6^1\rangle \\
CS & - \frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} & = -\frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_1^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_3\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle \otimes |S_4^1\rangle \\
CS &
\end{aligned}$$

$$\begin{aligned}
& -\frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_5^1\rangle + \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_8^1\rangle \\
& + \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}_{CS} & = -\frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_1^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_4\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_5\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_4^1\rangle \\
& - \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle \otimes |S_7^1\rangle + \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_8^1\rangle \\
& + \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} & = \frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle \otimes |S_1^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_3^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_5^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{6}}|C_1\rangle \otimes |S_6^1\rangle - \frac{\sqrt{10}}{9}|C_3\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_2\rangle \otimes |S_7^1\rangle + \frac{\sqrt{10}}{9}|C_4\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{9}|C_5\rangle \otimes |S_7^1\rangle \\
& - \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle \otimes |S_9^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} & = \frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle \otimes |S_1^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_3^1\rangle \\
& + \frac{\sqrt{5}}{9\sqrt{2}}|C_5\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_5^1\rangle - \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_5^1\rangle \\
& - \frac{\sqrt{5}}{6\sqrt{6}}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{10}}{9}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{9}|C_5\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{6\sqrt{6}}|C_1\rangle \otimes |S_7^1\rangle + \frac{\sqrt{10}}{9}|C_3\rangle \otimes |S_7^1\rangle \\
& + \frac{\sqrt{5}}{6\sqrt{6}}|C_4\rangle \otimes |S_8^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_5\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{6}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}_{CS} & = \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle \otimes |S_1^1\rangle + \frac{\sqrt{10}}{9}|C_5\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{3\sqrt{6}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{9\sqrt{2}}|C_4\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{3\sqrt{6}}|C_1\rangle \otimes |S_4^1\rangle \\
& - \frac{\sqrt{5}}{9\sqrt{2}}|C_3\rangle \otimes |S_4^1\rangle + \frac{2\sqrt{5}}{9}|C_5\rangle \otimes |S_5^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{5}}{9}|C_4\rangle \otimes |S_6^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_1\rangle \otimes |S_7^1\rangle \\
& - \frac{\sqrt{5}}{9}|C_3\rangle \otimes |S_7^1\rangle - \frac{\sqrt{5}}{6\sqrt{3}}|C_4\rangle \otimes |S_8^1\rangle + \frac{\sqrt{5}}{6\sqrt{3}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array}_{CS} & = \frac{\sqrt{5}}{3\sqrt{3}}|C_3\rangle \otimes |S_1^1\rangle - \frac{2}{3\sqrt{3}}|C_1\rangle \otimes |S_2^1\rangle - \frac{1}{9}|C_3\rangle \otimes |S_2^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{2}}{9}|C_3\rangle \otimes |S_3^1\rangle \\
& - \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle \otimes |S_4^1\rangle + \frac{\sqrt{2}}{9}|C_4\rangle \otimes |S_4^1\rangle - \frac{1}{9}|C_5\rangle \otimes |S_4^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle \otimes |S_5^1\rangle - \frac{\sqrt{2}}{9}|C_3\rangle \otimes |S_5^1\rangle \\
& - \frac{1}{3\sqrt{3}}|C_1\rangle \otimes |S_6^1\rangle - \frac{2}{9}|C_3\rangle \otimes |S_6^1\rangle + \frac{1}{3\sqrt{3}}|C_2\rangle \otimes |S_7^1\rangle + \frac{2}{9}|C_4\rangle \otimes |S_7^1\rangle - \frac{\sqrt{2}}{9}|C_5\rangle \otimes |S_7^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \quad CS$$

$$\begin{aligned}
& + \frac{1}{3\sqrt{3}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{3\sqrt{3}}|C_4\rangle \otimes |S_9^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle \otimes |S_9^1\rangle \\
& = \frac{\sqrt{5}}{3\sqrt{3}}|C_4\rangle \otimes |S_1^1\rangle - \frac{2}{3\sqrt{3}}|C_2\rangle \otimes |S_2^1\rangle - \frac{1}{9}|C_4\rangle \otimes |S_2^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{2}}{9}|C_4\rangle \otimes |S_3^1\rangle \\
& + \frac{1}{9}|C_5\rangle \otimes |S_3^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle + \frac{\sqrt{2}}{9}|C_3\rangle \otimes |S_4^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle \otimes |S_5^1\rangle - \frac{\sqrt{2}}{9}|C_4\rangle \otimes |S_5^1\rangle \\
& + \frac{1}{3\sqrt{3}}|C_2\rangle \otimes |S_6^1\rangle + \frac{2}{9}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{2}}{9}|C_5\rangle \otimes |S_6^1\rangle + \frac{1}{3\sqrt{3}}|C_1\rangle \otimes |S_7^1\rangle + \frac{2}{9}|C_3\rangle \otimes |S_7^1\rangle \\
& - \frac{1}{3\sqrt{3}}|C_4\rangle \otimes |S_8^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_5\rangle \otimes |S_8^1\rangle - \frac{1}{3\sqrt{3}}|C_3\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} \quad CS$$

$$\begin{aligned}
& = \frac{\sqrt{5}}{3\sqrt{3}}|C_5\rangle \otimes |S_1^1\rangle + \frac{2}{9}|C_5\rangle \otimes |S_2^1\rangle + \frac{2}{3\sqrt{3}}|C_2\rangle \otimes |S_3^1\rangle + \frac{1}{9}|C_4\rangle \otimes |S_3^1\rangle - \frac{2}{3\sqrt{3}}|C_1\rangle \otimes |S_4^1\rangle \\
& - \frac{1}{9}|C_3\rangle \otimes |S_4^1\rangle + \frac{2\sqrt{2}}{9}|C_5\rangle \otimes |S_5^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_2\rangle \otimes |S_6^1\rangle + \frac{\sqrt{2}}{9}|C_4\rangle \otimes |S_6^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_1\rangle \otimes |S_7^1\rangle \\
& - \frac{\sqrt{2}}{9}|C_3\rangle \otimes |S_7^1\rangle + \frac{\sqrt{2}}{3\sqrt{3}}|C_4\rangle \otimes |S_8^1\rangle - \frac{\sqrt{2}}{3\sqrt{3}}|C_3\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & \\ \hline 3 & \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} \quad CS$$

$$\begin{aligned}
& = -\frac{\sqrt{5}}{3\sqrt{3}}|C_5\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{3\sqrt{3}}|C_4\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{3\sqrt{3}}|C_3\rangle \otimes |S_4^1\rangle - \frac{2\sqrt{2}}{3\sqrt{15}}|C_5\rangle \otimes |S_5^1\rangle + \frac{2\sqrt{2}}{3\sqrt{15}}|C_4\rangle \otimes |S_6^1\rangle \\
& - \frac{2\sqrt{2}}{3\sqrt{15}}|C_3\rangle \otimes |S_7^1\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_8^1\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|} \hline 1 & 6 \\ \hline 2 & \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} \quad CS$$

$$= \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_5^1\rangle - \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_6^1\rangle + \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{5}}|C_2\rangle \otimes |S_8^1\rangle - \frac{1}{\sqrt{5}}|C_1\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \hline \end{array} \quad CS$$

$$= \frac{1}{\sqrt{2}}|C_1\rangle \otimes |S_8^1\rangle + \frac{1}{\sqrt{2}}|C_2\rangle \otimes |S_9^1\rangle$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & & \\ \hline \end{array} \quad CS$$

$$\begin{aligned}
& = -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_6^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_6^1\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_7^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_7^1\rangle + \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_8^1\rangle \\
& + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & & \\ \hline \end{array} \quad CS$$

$$\begin{aligned}
& = -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_5^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_5^1\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_7^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_7^1\rangle \\
& - \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_9^1\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_9^1\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_5^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_5^1\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_6^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_7^1\rangle \\
&\quad + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_8^1\rangle + \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_8^1\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_3^1\rangle - \frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_4^1\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_6^1\rangle \\
&\quad - \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_7^1\rangle - \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_1\rangle \otimes |S_2^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_3^1\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_4^1\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_4^1\rangle \\
&\quad - \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_5^1\rangle - \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_7^1\rangle + \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_8^1\rangle - \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_9^1\rangle \\
&\quad + \frac{1}{2\sqrt{5}}|C_5\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{10}}|C_2\rangle \otimes |S_2^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{2\sqrt{5}}|C_1\rangle \otimes |S_4^1\rangle \\
&\quad - \frac{1}{2\sqrt{5}}|C_2\rangle \otimes |S_5^1\rangle + \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_7^1\rangle - \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_8^1\rangle - \frac{1}{2\sqrt{5}}|C_5\rangle \otimes |S_8^1\rangle \\
&\quad - \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{2}}|C_1\rangle \otimes |S_1^1\rangle + \frac{1}{\sqrt{10}}|C_3\rangle \otimes |S_2^1\rangle - \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_3^1\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_4^1\rangle + \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_4^1\rangle \\
&\quad - \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_5^1\rangle + \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_6^1\rangle - \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_7^1\rangle - \frac{1}{2\sqrt{5}}|C_5\rangle \otimes |S_7^1\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_8^1\rangle \\
&\quad - \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_9^1\rangle \\
\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{2}}|C_2\rangle \otimes |S_1^1\rangle + \frac{1}{\sqrt{10}}|C_4\rangle \otimes |S_2^1\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_3^1\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_3^1\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_4^1\rangle \\
&\quad - \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_5^1\rangle - \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_6^1\rangle + \frac{1}{2\sqrt{5}}|C_5\rangle \otimes |S_6^1\rangle - \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_7^1\rangle - \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_8^1\rangle \\
&\quad - \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_9^1\rangle
\end{aligned}$$

3. S=2

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline 6 & & \\ \hline \end{array}_{CS} &= \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_5^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline 6 & & \\ \hline \end{array}_{CS} &= \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_3^2\rangle + \frac{1}{4}|C_1\rangle \otimes |S_4^2\rangle - \frac{1}{4}|C_2\rangle \otimes |S_5^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline 6 & & \\ \hline \end{array}_{CS} &= \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle \otimes |S_3^2\rangle - \frac{1}{4}|C_2\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle \otimes |S_4^2\rangle - \frac{1}{4}|C_1\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & & \\ \hline \end{array} &= -\frac{\sqrt{5}}{2\sqrt{2}}|C_1\rangle \otimes |S_2^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^2\rangle + \frac{1}{4}|C_3\rangle \otimes |S_4^2\rangle - \frac{1}{4}|C_4\rangle \otimes |S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_5^2\rangle \\
CS & \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline 6 & & \\ \hline \end{array} &= -\frac{\sqrt{5}}{2\sqrt{2}}|C_2\rangle \otimes |S_2^2\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_3^2\rangle - \frac{1}{4}|C_4\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_4^2\rangle - \frac{1}{4}|C_3\rangle \otimes |S_5^2\rangle \\
CS & \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 6 & \\ \hline 5 & & \\ \hline \end{array} &= \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_4^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^2\rangle \\
CS & \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \hline 5 & & \\ \hline \end{array} &= \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_3^2\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^2\rangle + \frac{\sqrt{3}}{4}|C_1\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{4}|C_2\rangle \otimes |S_5^2\rangle - \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_5^2\rangle \\
CS & \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 6 & \\ \hline 5 & & \\ \hline \end{array} &= \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_3^2\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{4}|C_2\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{2}}|C_5\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{4}|C_1\rangle \otimes |S_5^2\rangle \\
CS & \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & \\ \hline 4 & & \\ \hline \end{array} &= \frac{1}{\sqrt{2}}|C_3\rangle \otimes |S_2^2\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_3^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_3^2\rangle - \frac{1}{4\sqrt{5}}|C_1\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle \otimes |S_4^2\rangle \\
CS & \\
&+ \frac{1}{4\sqrt{5}}|C_2\rangle \otimes |S_5^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle \otimes |S_5^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & & \\ \hline \end{array} &= \frac{1}{\sqrt{2}}|C_4\rangle \otimes |S_2^2\rangle + \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_3^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_3^2\rangle + \frac{1}{4\sqrt{5}}|C_2\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle \\
CS & \\
&- \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_4^2\rangle + \frac{1}{4\sqrt{5}}|C_1\rangle \otimes |S_5^2\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 & \\ \hline 3 & & \\ \hline \end{array} &= \frac{1}{\sqrt{2}}|C_5\rangle \otimes |S_2^2\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_5\rangle \otimes |S_3^2\rangle - \frac{1}{2\sqrt{10}}|C_2\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{10}}|C_1\rangle \otimes |S_5^2\rangle \\
CS & \\
&+ \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 4 & \\ \hline 5 & & \\ \hline \end{array} &= -\frac{4}{5}|C_1\rangle \otimes |S_1^2\rangle - \frac{3\sqrt{3}}{10\sqrt{2}}|C_1\rangle \otimes |S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{4\sqrt{5}}|C_3\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_4\rangle \otimes |S_5^2\rangle \\
CS & \\
&+ \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \hline \end{array} &= -\frac{4}{5}|C_2\rangle \otimes |S_1^2\rangle - \frac{3\sqrt{3}}{10\sqrt{2}}|C_2\rangle \otimes |S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_3^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_5\rangle \otimes |S_4^2\rangle \\
CS & \\
&+ \frac{\sqrt{3}}{4\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 & \\ \hline 4 & & \\ \hline \end{array} &= -\frac{4}{5}|C_3\rangle \otimes |S_1^2\rangle + \frac{\sqrt{3}}{5\sqrt{2}}|C_3\rangle \otimes |S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_3^2\rangle - \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{4\sqrt{5}}|C_1\rangle \otimes |S_4^2\rangle \\
CS &
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_2\rangle \otimes |S_5^2\rangle + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_5^2\rangle - \frac{1}{2\sqrt{10}}|C_5\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} & = -\frac{4}{5}|C_4\rangle \otimes |S_1^2\rangle + \frac{\sqrt{3}}{5\sqrt{2}}|C_4\rangle \otimes |S_2^2\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_3^2\rangle - \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_3^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_2\rangle \otimes |S_4^2\rangle \\
CS & + \frac{1}{2\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{10}}|C_5\rangle \otimes |S_4^2\rangle + \frac{\sqrt{3}}{4\sqrt{5}}|C_1\rangle \otimes |S_5^2\rangle + \frac{1}{2\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & \\ \hline 3 & & \\ \hline \end{array} & = -\frac{4}{5}|C_5\rangle \otimes |S_1^2\rangle + \frac{\sqrt{3}}{5\sqrt{2}}|C_5\rangle \otimes |S_2^2\rangle + \frac{1}{\sqrt{10}}|C_5\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_4^2\rangle + \frac{1}{2\sqrt{10}}|C_4\rangle \otimes |S_4^2\rangle \\
CS & + \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_5^2\rangle - \frac{1}{2\sqrt{10}}|C_3\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} & = -\frac{3}{5}|C_1\rangle \otimes |S_1^2\rangle + \frac{\sqrt{6}}{5}|C_1\rangle \otimes |S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_3\rangle \otimes |S_3^2\rangle + \frac{1}{\sqrt{15}}|C_3\rangle \otimes |S_4^2\rangle - \frac{1}{\sqrt{15}}|C_4\rangle \otimes |S_5^2\rangle \\
CS & - \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline 6 & \\ \hline \end{array} & = -\frac{3}{5}|C_2\rangle \otimes |S_1^2\rangle + \frac{\sqrt{6}}{5}|C_2\rangle \otimes |S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_4\rangle \otimes |S_3^2\rangle - \frac{1}{\sqrt{15}}|C_4\rangle \otimes |S_4^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle \otimes |S_4^2\rangle \\
CS & - \frac{1}{\sqrt{15}}|C_3\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} & = -\frac{3}{5}|C_3\rangle \otimes |S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_3\rangle \otimes |S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_3^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_3^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |S_4^2\rangle \\
CS & + \frac{2}{3\sqrt{5}}|C_3\rangle \otimes |S_4^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle \otimes |S_5^2\rangle - \frac{2}{3\sqrt{5}}|C_4\rangle \otimes |S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_5\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} & = -\frac{3}{5}|C_4\rangle \otimes |S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_4\rangle \otimes |S_2^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_3^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_3^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle \otimes |S_4^2\rangle \\
CS & - \frac{2}{3\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_5\rangle \otimes |S_4^2\rangle - \frac{1}{\sqrt{15}}|C_1\rangle \otimes |S_5^2\rangle - \frac{2}{3\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline 6 & \\ \hline \end{array} & = -\frac{3}{5}|C_5\rangle \otimes |S_1^2\rangle - \frac{2\sqrt{2}}{5\sqrt{3}}|C_5\rangle \otimes |S_2^2\rangle - \frac{2\sqrt{2}}{3\sqrt{5}}|C_5\rangle \otimes |S_3^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle \\
CS & - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{3}}|C_3\rangle \otimes |S_2^2\rangle - \frac{2}{\sqrt{15}}|C_1\rangle \otimes |S_3^2\rangle - \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |S_3^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_4^2\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_4^2\rangle \\
&\quad - \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_5^2\rangle - \frac{1}{3\sqrt{5}}|C_5\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{3}}|C_4\rangle \otimes |S_2^2\rangle - \frac{2}{\sqrt{15}}|C_2\rangle \otimes |S_3^2\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |S_3^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_4^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle \\
&\quad + \frac{1}{3\sqrt{5}}|C_5\rangle \otimes |S_4^2\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_5^2\rangle + \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{3}}|C_5\rangle \otimes |S_2^2\rangle + \frac{2}{3\sqrt{5}}|C_5\rangle \otimes |S_3^2\rangle + \frac{2}{\sqrt{15}}|C_2\rangle \otimes |S_4^2\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |S_4^2\rangle - \frac{2}{\sqrt{15}}|C_1\rangle \otimes |S_5^2\rangle \\
&\quad - \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |S_5^2\rangle \\
\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{3}}|C_5\rangle \otimes |S_3^2\rangle + \frac{1}{\sqrt{3}}|C_4\rangle \otimes |S_4^2\rangle - \frac{1}{\sqrt{3}}|C_3\rangle \otimes |S_5^2\rangle
\end{aligned}$$

4. S=3

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}_{CS} &= |C_1\rangle \otimes |S_1^3\rangle, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}_{CS} &= |C_2\rangle \otimes |S_1^3\rangle, \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} &= |C_3\rangle \otimes |S_1^3\rangle, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}_{CS} &= |C_4\rangle \otimes |S_1^3\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}_{CS} &= |C_5\rangle \otimes |S_1^3\rangle
\end{aligned}$$

(B1)

Appendix C: CS coupling of q^5

1. S= $\frac{1}{2}$

$$\begin{aligned}
\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{2}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_1\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle - \frac{1}{2}|C_5\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_2\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2}|C_5\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle
\end{aligned}$$

$$\begin{aligned} \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_3\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2}|C_5\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}_{CS} &= \frac{1}{2}|C_4\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2}|C_5\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_5^{\frac{1}{2}}\rangle \end{aligned}$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} \quad CS = \frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array} \quad CS = \frac{\sqrt{3}}{2\sqrt{5}}|C_1\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{5}}|C_3\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\ + \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline 5 & & \\ \hline \end{array} \quad_{CS} = \frac{\sqrt{3}}{2\sqrt{5}}|C_2\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{5}}|C_4\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{5}}|C_5\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\ - \frac{\sqrt{3}}{2\sqrt{10}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{5}}|C_5\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} \quad CS = -\frac{1}{2}|C_1\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle - \frac{1}{2}|C_5\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} \quad CS = -\frac{1}{2}|C_2\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2}|C_5\rangle \otimes |S_4^{\frac{1}{2}}\rangle \\
- \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} \quad CS = \frac{1}{2}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{1}{2}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_4^{\frac{1}{2}}\rangle \\ + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}_{CS} = -\frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |S_4^{\frac{1}{2}}\rangle \\ + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |S_5^{\frac{1}{2}}\rangle$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array} CS &= -\frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{3}|C_3\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\
&+ \frac{\sqrt{2}}{3}|C_4\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{3}|C_5\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array} CS &= -\frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{3}|C_4\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{\sqrt{2}}{3}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{3}|C_5\rangle \otimes |S_2^{\frac{1}{2}}\rangle \\
&- \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{\sqrt{2}}{3}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{6}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array} CS &= \frac{2}{3}|C_5\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{3}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{3}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\
&- \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} CS &= -\frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{1}{3}|C_3\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\
&- \frac{1}{3}|C_4\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{3\sqrt{2}}|C_5\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_5^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{6}}|C_5\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} CS &= -\frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{1}{2\sqrt{3}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{3}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{3\sqrt{2}}|C_5\rangle \otimes |S_2^{\frac{1}{2}}\rangle \\
&- \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{1}{3}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{6}}|C_5\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{2\sqrt{3}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array} CS &= -\frac{\sqrt{2}}{3}|C_5\rangle \otimes |S_1^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{6}}|C_2\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle - \frac{1}{\sqrt{6}}|C_1\rangle \otimes |S_3^{\frac{1}{2}}\rangle + \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle \\
&- \frac{1}{\sqrt{6}}|C_4\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{1}{\sqrt{6}}|C_3\rangle \otimes |S_5^{\frac{1}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} CS &= \frac{\sqrt{2}}{\sqrt{15}}|C_5\rangle \otimes |S_1^{\frac{1}{2}}\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_4\rangle \otimes |S_2^{\frac{1}{2}}\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_3\rangle \otimes |S_3^{\frac{1}{2}}\rangle - \frac{\sqrt{3}}{\sqrt{10}}|C_2\rangle \otimes |S_4^{\frac{1}{2}}\rangle + \frac{\sqrt{3}}{\sqrt{10}}|C_1\rangle \otimes |S_5^{\frac{1}{2}}\rangle
\end{aligned}$$

2. $S = \frac{3}{2}$

$$\begin{aligned} \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{2\sqrt{2}}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_4\rangle \otimes |S_4^{\frac{3}{2}}\rangle \\ \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{2\sqrt{2}}|C_1\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{1}{4}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{1}{4}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_3\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|C_5\rangle \otimes |S_4^{\frac{3}{2}}\rangle \end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{6}}|C_4\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{4\sqrt{3}}|C_2\rangle \otimes |S_3^{\frac{3}{2}}\rangle \\
&+ \frac{\sqrt{5}}{6}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_5\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{4\sqrt{3}}|C_1\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{6}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{6}}|C_5\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{3\sqrt{2}}|C_5\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{\sqrt{5}}{2\sqrt{6}}|C_2\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{5}}{6\sqrt{2}}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle \\
&+ \frac{\sqrt{5}}{2\sqrt{6}}|C_1\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{\sqrt{5}}{6\sqrt{2}}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle
\end{aligned}$$

$$\begin{aligned}
\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{3}}|C_3\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{2}{\sqrt{15}}|C_1\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_3^{\frac{3}{2}}\rangle \\
&+ \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{1}{3\sqrt{5}}|C_5\rangle \otimes |S_4^{\frac{3}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{3}}|C_4\rangle \otimes |S_1^{\frac{3}{2}}\rangle + \frac{2}{\sqrt{15}}|C_2\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |S_2^{\frac{3}{2}}\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |S_3^{\frac{3}{2}}\rangle \\
&- \frac{\sqrt{2}}{3\sqrt{5}}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{3\sqrt{5}}|C_5\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |S_4^{\frac{3}{2}}\rangle - \frac{\sqrt{2}}{3\sqrt{5}}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array}_{CS} &= -\frac{1}{\sqrt{3}}|C_5\rangle \otimes |S_1^{\frac{3}{2}}\rangle - \frac{2}{3\sqrt{5}}|C_5\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{2}{\sqrt{15}}|C_2\rangle \otimes |S_3^{\frac{3}{2}}\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle \\
&+ \frac{2}{\sqrt{15}}|C_1\rangle \otimes |S_4^{\frac{3}{2}}\rangle + \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle \\
\begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}_{CS} &= \frac{1}{\sqrt{3}}|C_5\rangle \otimes |S_2^{\frac{3}{2}}\rangle - \frac{1}{\sqrt{3}}|C_4\rangle \otimes |S_3^{\frac{3}{2}}\rangle + \frac{1}{\sqrt{3}}|C_3\rangle \otimes |S_4^{\frac{3}{2}}\rangle
\end{aligned}$$

3. $S=\frac{5}{2}$

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}_{CS} = |C_1\rangle \otimes |S_1^{\frac{5}{2}}\rangle, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \hline \end{array}_{CS} = |C_2\rangle \otimes |S_1^{\frac{5}{2}}\rangle, \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline \end{array}_{CS} = |C_4\rangle \otimes |S_1^{\frac{5}{2}}\rangle, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array}_{CS} = |C_4\rangle \otimes |S_1^{\frac{5}{2}}\rangle, \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \hline \end{array}_{CS} = |C_5\rangle \otimes |S_1^{\frac{5}{2}}\rangle. \quad (C1)$$

Appendix D: Spin basis transformation : $q^5 \rightarrow q^5 Q$

- $S=0$: $|S_1^{\frac{1}{2}}\rangle \rightarrow |S_1^0\rangle$, $|S_2^{\frac{1}{2}}\rangle \rightarrow |S_2^0\rangle$, $|S_3^{\frac{1}{2}}\rangle \rightarrow |S_3^0\rangle$, $|S_4^{\frac{1}{2}}\rangle \rightarrow |S_4^0\rangle$, $|S_5^{\frac{1}{2}}\rangle \rightarrow |S_5^0\rangle$

- S=1 : $|S_1^{\frac{1}{2}}\rangle \rightarrow |S_5^1\rangle$, $|S_2^{\frac{1}{2}}\rangle \rightarrow |S_6^1\rangle$, $|S_3^{\frac{1}{2}}\rangle \rightarrow |S_7^1\rangle$, $|S_4^{\frac{1}{2}}\rangle \rightarrow |S_8^1\rangle$, $|S_5^{\frac{1}{2}}\rangle \rightarrow |S_9^1\rangle$,
 $|S_1^{\frac{3}{2}}\rangle \rightarrow |S_1^1\rangle$, $|S_2^{\frac{3}{2}}\rangle \rightarrow |S_2^1\rangle$, $|S_3^{\frac{3}{2}}\rangle \rightarrow |S_3^1\rangle$, $|S_4^{\frac{3}{2}}\rangle \rightarrow |S_4^1\rangle$
- S=2 : $|S_1^{\frac{3}{2}}\rangle \rightarrow |S_2^2\rangle$, $|S_2^{\frac{3}{2}}\rangle \rightarrow |S_3^2\rangle$, $|S_3^{\frac{3}{2}}\rangle \rightarrow |S_4^2\rangle$, $|S_4^{\frac{3}{2}}\rangle \rightarrow |S_5^2\rangle$, $|S_1^{\frac{5}{2}}\rangle \rightarrow |S_1^2\rangle$
- S=3 : $|S_1^{\frac{5}{2}}\rangle \rightarrow |S_1^3\rangle$

Appendix E: Flavor, color and spin state of q^5Q

$$\begin{aligned}
\psi_{1,S=0} = & -\frac{\sqrt{3}}{4\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{4\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{\sqrt{3}}{4\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{4\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
& + \frac{\sqrt{3}}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{1}{8}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{4\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle \\
& + \frac{1}{8}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle + \frac{\sqrt{3}}{8}|C_1\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle + \frac{1}{8}|C_3\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle - \frac{\sqrt{3}}{8}|C_2\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle \\
& - \frac{1}{8}|C_4\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle - \frac{1}{4\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle + \frac{1}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle - \frac{\sqrt{3}}{8}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle \\
& - \frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + \frac{1}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle - \frac{\sqrt{3}}{8}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle + \frac{1}{8}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle \\
& - \frac{\sqrt{3}}{8}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle - \frac{1}{8}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle + \frac{\sqrt{3}}{8}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle - \frac{\sqrt{3}}{4\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle \\
\psi_{2,S=0} = & -\frac{\sqrt{3}}{4}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{4}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{\sqrt{3}}{4}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{4}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
& + \frac{1}{4}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{1}{4}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle + \frac{\sqrt{3}}{4}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + \frac{\sqrt{3}}{4}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle \\
\psi_{3,S=0} = & -\frac{1}{4\sqrt{6}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{12\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{4\sqrt{6}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{12\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
& - \frac{1}{8\sqrt{3}}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle + \frac{7}{24}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{12\sqrt{2}}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{8\sqrt{3}}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle \\
& + \frac{7}{24}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle - \frac{1}{8\sqrt{3}}|C_1\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle + \frac{7}{24}|C_3\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle + \frac{1}{8\sqrt{3}}|C_2\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle \\
& - \frac{7}{24}|C_4\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle - \frac{1}{12\sqrt{2}}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle - \frac{3}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle + \frac{1}{8\sqrt{3}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle \\
& - \frac{1}{4\sqrt{6}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle - \frac{3}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle + \frac{1}{8\sqrt{3}}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^0\rangle - \frac{3}{8}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle \\
& + \frac{1}{8\sqrt{3}}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^0\rangle + \frac{3}{8}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle - \frac{1}{8\sqrt{3}}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle - \frac{1}{4\sqrt{6}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^0\rangle \\
\psi_{4,S=0} = & -\frac{1}{2\sqrt{3}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{6}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^0\rangle + \frac{1}{2\sqrt{3}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle - \frac{1}{6}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^0\rangle \\
& - \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{6}|C_5\rangle \otimes |F_1\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle \\
& - \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^0\rangle - \frac{1}{2\sqrt{6}}|C_1\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle - \frac{1}{3\sqrt{2}}|C_3\rangle \otimes |F_1\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{6}}|C_2\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle \\
& + \frac{1}{3\sqrt{2}}|C_4\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle - \frac{1}{6}|C_5\rangle \otimes |F_2\rangle \otimes |S_3^0\rangle + \frac{1}{2\sqrt{6}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle - \frac{1}{2\sqrt{3}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^0\rangle
\end{aligned}$$

[illegible]

$$-\frac{1}{2}|C_5\rangle\otimes|F_1\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_2\rangle\otimes|S_3^1\rangle-\frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle$$

$$+\frac{1}{8\sqrt{2}}|C_3\rangle\otimes|F_1\rangle\otimes|S_4^1\rangle+\frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{8\sqrt{2}}|C_4\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle-\frac{1}{2}|C_5\rangle\otimes|F_2\rangle\otimes|S_4^1\rangle$$

$$\begin{aligned}
\psi_{6,S=1} &= \frac{\sqrt{3}}{4\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^1\rangle - \frac{\sqrt{3}}{4\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^1\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle - \frac{1}{4\sqrt{10}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle \\
&+ \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle + \frac{1}{4\sqrt{10}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle - \frac{3}{8\sqrt{5}}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle \\
&- \frac{1}{\sqrt{10}}|C_5\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle - \frac{3}{8\sqrt{5}}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle \\
&- \frac{3}{8\sqrt{5}}|C_3\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle - \frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle + \frac{3}{8\sqrt{5}}|C_4\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle \\
\psi_{7,S=1} &= \frac{3}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^1\rangle + \frac{1}{4\sqrt{3}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^1\rangle - \frac{3}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^1\rangle - \frac{1}{4\sqrt{3}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^1\rangle \\
&+ \frac{\sqrt{5}}{8\sqrt{3}}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{6}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle - \frac{\sqrt{5}}{8\sqrt{3}}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle + \frac{\sqrt{5}}{6}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle \\
&+ \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{6}|C_5\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle \\
&- \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle - \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle \\
&+ \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle + \frac{\sqrt{5}}{6}|C_5\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle
\end{aligned} \tag{E1}$$

$$\begin{aligned} \psi_{8,S=1} = & -\frac{1}{\sqrt{6}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^1\rangle + \frac{1}{\sqrt{6}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^1\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle + \frac{1}{3\sqrt{10}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^1\rangle \\ & -\frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle - \frac{1}{3\sqrt{10}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^1\rangle + \frac{1}{\sqrt{15}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle \\ & -\frac{1}{3\sqrt{10}}|C_5\rangle \otimes |F_1\rangle \otimes |S_3^1\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle - \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^1\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle \\ & -\frac{1}{3\sqrt{5}}|C_3\rangle \otimes |F_1\rangle \otimes |S_4^1\rangle - \frac{1}{\sqrt{15}}|C_2\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle - \frac{1}{3\sqrt{10}}|C_5\rangle \otimes |F_2\rangle \otimes |S_4^1\rangle \end{aligned}$$

$$\begin{aligned} \psi_{1,S=2} = & \frac{\sqrt{5}}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^2\rangle - \frac{\sqrt{5}}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^2\rangle + \frac{\sqrt{3}}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle - \frac{1}{4}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle \\ & - \frac{\sqrt{3}}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle + \frac{1}{4}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle - \frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{1}{8\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle \\ & - \frac{1}{2}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle + \frac{1}{8\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle - \frac{\sqrt{3}}{8\sqrt{2}}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle \\ & + \frac{1}{8\sqrt{2}}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle + \frac{\sqrt{3}}{8\sqrt{2}}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle - \frac{1}{8\sqrt{2}}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle - \frac{1}{2}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle \end{aligned}$$

$$\begin{aligned} \psi_{2,S=2} = & \frac{\sqrt{3}}{4\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^2\rangle - \frac{\sqrt{3}}{4\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^2\rangle - \frac{\sqrt{3}}{2\sqrt{10}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle - \frac{1}{4\sqrt{10}}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle \\ & + \frac{\sqrt{3}}{2\sqrt{10}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle + \frac{1}{4\sqrt{10}}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle - \frac{3}{8\sqrt{5}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\sqrt{10}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle - \frac{3}{8\sqrt{5}}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle + \frac{3\sqrt{3}}{8\sqrt{5}}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle \\
& -\frac{3}{8\sqrt{5}}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle - \frac{3\sqrt{3}}{8\sqrt{5}}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle + \frac{3}{8\sqrt{5}}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle - \frac{1}{\sqrt{10}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle \\
\psi_{3,S=2} = & \frac{3}{8}|C_2\rangle \otimes |F_1\rangle \otimes |S_2^2\rangle + \frac{1}{4\sqrt{3}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^2\rangle - \frac{3}{8}|C_1\rangle \otimes |F_2\rangle \otimes |S_2^2\rangle - \frac{1}{4\sqrt{3}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^2\rangle \\
& + \frac{\sqrt{5}}{8\sqrt{3}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle - \frac{\sqrt{5}}{6}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle - \frac{\sqrt{5}}{8\sqrt{3}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle + \frac{\sqrt{5}}{6}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle \\
& + \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{\sqrt{5}}{6}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle \\
& - \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle + \frac{\sqrt{5}}{8\sqrt{6}}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle - \frac{\sqrt{5}}{24\sqrt{2}}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle - \frac{\sqrt{5}}{8\sqrt{6}}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle \\
& + \frac{\sqrt{5}}{24\sqrt{2}}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle + \frac{\sqrt{5}}{6}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle \tag{E2} \\
\psi_{4,S=2} = & -\frac{1}{\sqrt{6}}|C_4\rangle \otimes |F_1\rangle \otimes |S_2^2\rangle + \frac{1}{\sqrt{6}}|C_3\rangle \otimes |F_2\rangle \otimes |S_2^2\rangle + \frac{\sqrt{2}}{\sqrt{15}}|C_2\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle + \frac{1}{3\sqrt{10}}|C_4\rangle \otimes |F_1\rangle \otimes |S_3^2\rangle \\
& - \frac{\sqrt{2}}{\sqrt{15}}|C_1\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle - \frac{1}{3\sqrt{10}}|C_3\rangle \otimes |F_2\rangle \otimes |S_3^2\rangle + \frac{1}{\sqrt{15}}|C_2\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle - \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle \\
& - \frac{1}{3\sqrt{10}}|C_5\rangle \otimes |F_1\rangle \otimes |S_4^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle - \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |F_2\rangle \otimes |S_4^2\rangle + \frac{1}{\sqrt{15}}|C_1\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle \\
& - \frac{1}{3\sqrt{5}}|C_3\rangle \otimes |F_1\rangle \otimes |S_5^2\rangle - \frac{1}{\sqrt{15}}|C_2\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle + \frac{1}{3\sqrt{5}}|C_4\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle - \frac{1}{3\sqrt{10}}|C_5\rangle \otimes |F_2\rangle \otimes |S_5^2\rangle \\
\psi_{5,S=2} = & -\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^2\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^2\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^2\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^2\rangle \\
\psi_{1,S=3} = & -\frac{\sqrt{3}}{2\sqrt{2}}|C_2\rangle \otimes |F_1\rangle \otimes |S_1^3\rangle + \frac{1}{2\sqrt{2}}|C_4\rangle \otimes |F_1\rangle \otimes |S_1^3\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|C_1\rangle \otimes |F_2\rangle \otimes |S_1^3\rangle - \frac{1}{2\sqrt{2}}|C_3\rangle \otimes |F_2\rangle \otimes |S_1^3\rangle
\end{aligned}$$

Appendix F: Matrix elements of $\lambda_i\lambda_j$ and $\lambda_i\lambda_j\sigma_i \cdot \sigma_j$

We represent the matrix elements of $\lambda_i\lambda_j = \langle\psi|\lambda_i\lambda_j|\psi\rangle$ and $\lambda_i\lambda_j\sigma_i \cdot \sigma_j = \langle\psi|\lambda_i\lambda_j\sigma_i \cdot \sigma_j|\psi\rangle$ only for $S=0,1,2,3$.

1. $S=0$

$$\begin{aligned}
\lambda_1\lambda_2 = & \begin{pmatrix} -\frac{7}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} \\ \frac{1}{\sqrt{2}} & -\frac{5}{3} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ -\frac{1}{6} & \frac{1}{3\sqrt{2}} & -\frac{13}{18} & -\frac{\sqrt{2}}{9} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{\sqrt{2}}{9} & -\frac{10}{9} \end{pmatrix}, \quad \lambda_1\lambda_4 = \begin{pmatrix} -\frac{11}{12} & -\frac{1}{2\sqrt{2}} & -\frac{1}{12} & \frac{\sqrt{2}}{3} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{6\sqrt{2}} & -\frac{1}{3} \\ -\frac{1}{12} & -\frac{1}{6\sqrt{2}} & -\frac{41}{36} & \frac{2\sqrt{2}}{9} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2\sqrt{2}}{9} & -\frac{10}{9} \end{pmatrix}, \quad \lambda_4\lambda_5 = \begin{pmatrix} -\frac{5}{3} & 0 & 1 & -\sqrt{2} \\ 0 & -\frac{8}{3} & 0 & 0 \\ 1 & 0 & -\frac{5}{3} & -\sqrt{2} \\ -\sqrt{2} & 0 & -\sqrt{2} & -\frac{2}{3} \end{pmatrix}, \\
\lambda_1\lambda_6 = & \begin{pmatrix} -\frac{7}{6} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{3\sqrt{2}} & -\frac{29}{18} & -\frac{2\sqrt{2}}{9} \\ 0 & -\frac{2}{3} & -\frac{2\sqrt{2}}{9} & -\frac{8}{9} \end{pmatrix}, \quad \lambda_5\lambda_6 = \begin{pmatrix} -\frac{11}{12} & \frac{3}{2\sqrt{2}} & -\frac{3}{4} & 0 \\ \frac{3}{2\sqrt{2}} & -\frac{7}{6} & \frac{1}{2\sqrt{2}} & 1 \\ -\frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{2}}{3} \\ 0 & 1 & \frac{\sqrt{2}}{3} & -\frac{4}{3} \end{pmatrix}. \tag{F1}
\end{aligned}$$

$$\begin{aligned}
\lambda_1 \lambda_2 \sigma_1 \sigma_2 &= \begin{pmatrix} \frac{5}{6} & -\frac{5}{3\sqrt{2}} & \frac{1}{18} & \frac{\sqrt{2}}{9} \\ -\frac{5}{3\sqrt{2}} & 1 & -\frac{5}{9\sqrt{2}} & -\frac{10}{9} \\ \frac{1}{18} & -\frac{5}{9\sqrt{2}} & \frac{37}{54} & \frac{\sqrt{2}}{27} \\ \frac{\sqrt{2}}{9} & -\frac{10}{9} & \frac{\sqrt{2}}{27} & \frac{22}{27} \end{pmatrix}, \quad \lambda_1 \lambda_4 \sigma_1 \sigma_4 = \begin{pmatrix} \frac{11}{4} & \frac{5}{6\sqrt{2}} & \frac{1}{36} & -\frac{\sqrt{2}}{9} \\ \frac{5}{6\sqrt{2}} & \frac{23}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} \\ \frac{1}{36} & \frac{5}{18\sqrt{2}} & \frac{17}{108} & -\frac{2\sqrt{2}}{27} \\ -\frac{\sqrt{2}}{9} & \frac{5}{9} & -\frac{2\sqrt{2}}{27} & -\frac{50}{27} \end{pmatrix}, \\
\lambda_4 \lambda_5 \sigma_4 \sigma_5 &= \begin{pmatrix} -3 & 0 & -\frac{1}{3} & \frac{\sqrt{2}}{3} \\ 0 & -\frac{8}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & -3 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{10}{3} \end{pmatrix}, \quad \lambda_1 \lambda_6 \sigma_1 \sigma_6 = \begin{pmatrix} \frac{5}{6} & \frac{5}{3\sqrt{2}} & -\frac{1}{6} & 0 \\ \frac{5}{3\sqrt{2}} & \frac{31}{9} & \frac{5}{9\sqrt{2}} & \frac{10}{9} \\ -\frac{1}{6} & \frac{5}{9\sqrt{2}} & -\frac{43}{54} & -\frac{22\sqrt{2}}{27} \\ 0 & \frac{10}{9} & -\frac{22\sqrt{2}}{27} & -\frac{4}{27} \end{pmatrix}, \\
\lambda_5 \lambda_6 \sigma_5 \sigma_6 &= \begin{pmatrix} \frac{11}{4} & -\frac{5}{2\sqrt{2}} & \frac{1}{4} & 0 \\ -\frac{5}{2\sqrt{2}} & \frac{17}{6} & -\frac{5}{6\sqrt{2}} & -\frac{5}{3} \\ \frac{1}{4} & -\frac{5}{6\sqrt{2}} & -\frac{101}{36} & \frac{11\sqrt{2}}{9} \\ 0 & -\frac{5}{3} & \frac{11\sqrt{2}}{9} & \frac{20}{9} \end{pmatrix}.
\end{aligned} \tag{F2}$$

2. S=1

$$\begin{aligned}
\lambda_1 \lambda_2 &= \begin{pmatrix} -\frac{7}{6} & \frac{1}{\sqrt{2}} & -\frac{1}{6} & -\frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{5}{3} & \frac{1}{3\sqrt{2}} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3\sqrt{2}} & -\frac{13}{18} & -\frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{\sqrt{2}}{9} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{3} & -\sqrt{\frac{2}{5}} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & -\frac{16}{15} & \frac{\sqrt{2}}{3} & -\frac{2}{15} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{3} & \frac{\sqrt{2}}{3} & -\frac{11}{9} & \frac{\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{2}{5}}}{3} & -\frac{2}{15} & \frac{\sqrt{2}}{9} & -\frac{32}{45} \end{pmatrix}, \\
\lambda_1 \lambda_4 &= \begin{pmatrix} -\frac{11}{12} & -\frac{1}{2\sqrt{2}} & -\frac{1}{12} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2} & -\frac{1}{6\sqrt{2}} & -\frac{1}{3} & 0 & 0 & 0 & 0 \\ -\frac{1}{12} & -\frac{1}{6\sqrt{2}} & -\frac{41}{36} & \frac{2\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2\sqrt{2}}{9} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{13}{24} & \frac{3}{4\sqrt{10}} & -\frac{5\sqrt{5}}{24} & \frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & \frac{3}{4\sqrt{10}} & -\frac{19}{20} & -\frac{7}{12\sqrt{2}} & \frac{4}{15} \\ 0 & 0 & 0 & 0 & -\frac{5\sqrt{5}}{24} & -\frac{7}{12\sqrt{2}} & -\frac{67}{72} & \frac{\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{3} & \frac{4}{15} & \frac{\sqrt{2}}{9} & -\frac{56}{45} \end{pmatrix}, \\
\lambda_4 \lambda_5 &= \begin{pmatrix} -\frac{5}{3} & 0 & 1 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{5}{3} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{29}{12} & \frac{3}{2\sqrt{10}} & \frac{\sqrt{5}}{4} & -\sqrt{\frac{2}{5}} \\ 0 & 0 & 0 & 0 & \frac{3}{2\sqrt{10}} & -\frac{53}{30} & \frac{3}{2\sqrt{2}} & -\frac{6}{5} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{5}}{4} & \frac{3}{2\sqrt{2}} & -\frac{17}{12} & -\sqrt{2} \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{2}{5}} & -\frac{6}{5} & -\sqrt{2} & -\frac{16}{15} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\lambda_1 \lambda_6 = & \begin{pmatrix} -\frac{7}{6} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{3\sqrt{2}} & -\frac{29}{18} & -\frac{2\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & -\frac{2\sqrt{2}}{9} & -\frac{8}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{11}{12} & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{13}{10} & -\frac{1}{6\sqrt{2}} & -\frac{4}{15} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{4} & -\frac{1}{6\sqrt{2}} & -\frac{37}{36} & -\frac{4\sqrt{2}}{9} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{15} & -\frac{4\sqrt{2}}{9} & -\frac{64}{45} \end{pmatrix}, \\
\lambda_5 \lambda_6 = & \begin{pmatrix} -\frac{11}{12} & \frac{3}{2\sqrt{2}} & -\frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2\sqrt{2}} & -\frac{7}{6} & \frac{1}{2\sqrt{2}} & 1 & 0 & 0 & 0 & 0 \\ -\frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{2}}{3} & -\frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{31}{24} & -\frac{3\sqrt{5}}{4} & \frac{3\sqrt{5}}{8} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3\sqrt{\frac{5}{2}}}{4} & -\frac{43}{60} & \frac{1}{4\sqrt{2}} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3\sqrt{5}}{8} & \frac{1}{4\sqrt{2}} & -\frac{9}{8} & \frac{2\sqrt{2}}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & \frac{2\sqrt{2}}{3} & -\frac{8}{15} \end{pmatrix}.
\end{aligned} \tag{F3}$$

$$\begin{aligned}
\lambda_1 \lambda_2 \sigma_1 \sigma_2 = & \begin{pmatrix} \frac{5}{6} & -\frac{5}{3\sqrt{2}} & \frac{1}{18} & \frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ -\frac{5}{3\sqrt{2}} & 1 & -\frac{5}{9\sqrt{2}} & -\frac{10}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{18} & -\frac{5}{9\sqrt{2}} & \frac{37}{54} & \frac{\sqrt{2}}{27} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{9} & -\frac{10}{9} & \frac{\sqrt{2}}{27} & \frac{22}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{6} & \frac{7}{3\sqrt{10}} & -\frac{11\sqrt{5}}{18} & \frac{\sqrt{\frac{2}{5}}}{9} \\ 0 & 0 & 0 & 0 & \frac{7}{3\sqrt{10}} & \frac{3}{5} & -\frac{7}{9\sqrt{2}} & \frac{34}{45} \\ 0 & 0 & 0 & 0 & -\frac{11\sqrt{5}}{18} & -\frac{7}{9\sqrt{2}} & \frac{19}{54} & -\frac{\sqrt{2}}{27} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{9} & \frac{34}{45} & -\frac{\sqrt{2}}{27} & -\frac{16}{135} \end{pmatrix}, \\
\lambda_1 \lambda_4 \sigma_1 \sigma_4 = & \begin{pmatrix} \frac{11}{4} & \frac{5}{6\sqrt{2}} & \frac{1}{36} & -\frac{\sqrt{2}}{9} & 0 & 0 & 0 & 0 \\ \frac{5}{6\sqrt{2}} & \frac{23}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{5}{18\sqrt{2}} & \frac{17}{108} & -\frac{2\sqrt{2}}{27} & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{9} & \frac{5}{9} & -\frac{2\sqrt{2}}{27} & -\frac{50}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{15}{8} & -\frac{13}{12\sqrt{10}} & \frac{23\sqrt{5}}{72} & -\frac{\sqrt{\frac{2}{5}}}{9} \\ 0 & 0 & 0 & 0 & -\frac{13}{12\sqrt{10}} & \frac{31}{36} & \frac{17}{36\sqrt{2}} & -\frac{4}{9} \\ 0 & 0 & 0 & 0 & \frac{23\sqrt{5}}{72} & \frac{17}{36\sqrt{2}} & -\frac{71}{216} & -\frac{\sqrt{2}}{27} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{\frac{2}{5}}}{9} & -\frac{4}{9} & -\frac{\sqrt{2}}{27} & -\frac{56}{27} \end{pmatrix}, \\
\lambda_4 \lambda_5 \sigma_4 \sigma_5 = & \begin{pmatrix} -3 & 0 & -\frac{1}{3} & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -3 & \frac{\sqrt{2}}{3} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{2}}{3} & -\frac{10}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{11}{4} & -\frac{1}{2\sqrt{10}} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{\frac{2}{5}}}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{10}} & -\frac{89}{30} & -\frac{1}{2\sqrt{2}} & \frac{2}{5} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{5}}{12} & -\frac{1}{2\sqrt{2}} & -\frac{37}{12} & \frac{\sqrt{2}}{3} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\frac{2}{5}}}{3} & \frac{2}{5} & \frac{\sqrt{2}}{3} & -\frac{16}{5} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\lambda_1 \lambda_6 \sigma_1 \sigma_6 = & \begin{pmatrix} -\frac{5}{18} & -\frac{5}{9\sqrt{2}} & \frac{1}{18} & 0 & -\frac{16}{9} & -\frac{2\sqrt{10}}{9} & -\frac{4\sqrt{5}}{9} & 0 \\ -\frac{5}{9\sqrt{2}} & -\frac{31}{27} & -\frac{5}{27\sqrt{2}} & -\frac{10}{27} & -\frac{2\sqrt{2}}{3} & \frac{56}{27\sqrt{5}} & \frac{2\sqrt{10}}{27} & \frac{32}{27\sqrt{5}} \\ \frac{1}{18} & -\frac{5}{27\sqrt{2}} & \frac{43}{162} & \frac{22\sqrt{2}}{81} & \frac{4}{9} & -\frac{22\sqrt{\frac{2}{5}}}{27} & -\frac{88\sqrt{5}}{81} & -\frac{32\sqrt{\frac{2}{5}}}{81} \\ 0 & -\frac{10}{27} & \frac{22\sqrt{2}}{81} & \frac{4}{81} & 0 & \frac{56}{27\sqrt{5}} & -\frac{8\sqrt{10}}{81} & -\frac{464}{81\sqrt{5}} \\ -\frac{16}{9} & -\frac{2\sqrt{2}}{3} & \frac{4}{9} & 0 & \frac{95}{36} & -\frac{5\sqrt{\frac{5}{2}}}{6} & \frac{7\sqrt{5}}{36} & 0 \\ -\frac{2\sqrt{10}}{9} & \frac{56}{27\sqrt{5}} & -\frac{22\sqrt{\frac{2}{5}}}{27} & \frac{56}{27\sqrt{5}} & -\frac{5\sqrt{\frac{5}{2}}}{6} & \frac{5}{54} & \frac{47}{54\sqrt{2}} & -\frac{20}{27} \\ -\frac{4\sqrt{5}}{9} & \frac{2\sqrt{10}}{27} & -\frac{88\sqrt{5}}{81} & -\frac{8\sqrt{10}}{81} & \frac{7\sqrt{5}}{36} & \frac{47}{54\sqrt{2}} & \frac{307}{324} & \frac{28\sqrt{2}}{81} \\ 0 & \frac{32}{27\sqrt{5}} & -\frac{32\sqrt{\frac{2}{5}}}{81} & -\frac{464}{81\sqrt{5}} & 0 & -\frac{20}{27} & \frac{28\sqrt{2}}{81} & \frac{80}{81} \end{pmatrix}, \\
\lambda_5 \lambda_6 \sigma_5 \sigma_6 = & \begin{pmatrix} -\frac{11}{12} & \frac{5}{6\sqrt{2}} & -\frac{1}{12} & 0 & -\frac{4}{3} & \frac{\sqrt{10}}{3} & \frac{2\sqrt{5}}{3} & 0 \\ \frac{5}{6\sqrt{2}} & -\frac{17}{18} & \frac{5}{18\sqrt{2}} & \frac{5}{9} & \sqrt{2} & \frac{32}{9\sqrt{5}} & -\frac{\sqrt{10}}{9} & -\frac{16}{9\sqrt{5}} \\ -\frac{1}{12} & \frac{5}{18\sqrt{2}} & \frac{101}{108} & -\frac{11\sqrt{2}}{27} & -\frac{2}{3} & \frac{11\sqrt{\frac{2}{5}}}{9} & \frac{8\sqrt{5}}{27} & \frac{16\sqrt{\frac{2}{5}}}{27} \\ 0 & \frac{5}{9} & -\frac{11\sqrt{2}}{27} & -\frac{20}{27} & 0 & -\frac{28}{9\sqrt{5}} & \frac{4\sqrt{10}}{27} & -\frac{128}{27\sqrt{5}} \\ -\frac{4}{3} & \sqrt{2} & -\frac{2}{3} & 0 & \frac{65}{24} & \frac{5\sqrt{\frac{5}{2}}}{4} & -\frac{7\sqrt{5}}{24} & 0 \\ \frac{\sqrt{10}}{3} & \frac{32}{9\sqrt{5}} & \frac{11\sqrt{\frac{2}{5}}}{9} & -\frac{28}{9\sqrt{5}} & \frac{5\sqrt{\frac{5}{2}}}{4} & -\frac{5}{36} & -\frac{47}{36\sqrt{2}} & \frac{10}{9} \\ \frac{2\sqrt{5}}{3} & -\frac{\sqrt{10}}{9} & \frac{8\sqrt{5}}{27} & \frac{4\sqrt{10}}{27} & -\frac{7\sqrt{5}}{24} & -\frac{47}{36\sqrt{2}} & \frac{557}{216} & -\frac{14\sqrt{2}}{27} \\ 0 & -\frac{16}{9\sqrt{5}} & \frac{16\sqrt{\frac{2}{5}}}{27} & -\frac{128}{27\sqrt{5}} & 0 & \frac{10}{9} & -\frac{14\sqrt{2}}{27} & -\frac{40}{27} \end{pmatrix}. \tag{F4}
\end{aligned}$$

3. S=2

$$\begin{aligned}
\lambda_1 \lambda_2 = & \begin{pmatrix} -\frac{5}{3} & -\sqrt{\frac{2}{5}} & \frac{\sqrt{5}}{3} & -\frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ -\sqrt{\frac{2}{5}} & -\frac{16}{15} & \frac{\sqrt{2}}{3} & -\frac{2}{15} & 0 \\ \frac{\sqrt{5}}{3} & \frac{\sqrt{2}}{3} & -\frac{11}{9} & \frac{\sqrt{2}}{9} & 0 \\ -\frac{\sqrt{\frac{2}{5}}}{3} & -\frac{2}{15} & \frac{\sqrt{2}}{9} & -\frac{32}{45} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}, \quad \lambda_1 \lambda_4 = \begin{pmatrix} -\frac{13}{24} & \frac{3}{4\sqrt{10}} & -\frac{5\sqrt{5}}{24} & \frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ \frac{3}{4\sqrt{10}} & -\frac{19}{20} & -\frac{7}{12\sqrt{2}} & \frac{4}{15} & 0 \\ -\frac{5\sqrt{5}}{24} & -\frac{7}{12\sqrt{2}} & -\frac{67}{72} & \frac{\sqrt{2}}{9} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{3} & \frac{4}{15} & \frac{\sqrt{2}}{9} & -\frac{56}{45} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \\
\lambda_4 \lambda_5 = & \begin{pmatrix} -\frac{29}{12} & \frac{3}{2\sqrt{10}} & \frac{\sqrt{5}}{4} & -\sqrt{\frac{2}{5}} & 0 \\ \frac{3}{2\sqrt{10}} & -\frac{53}{30} & \frac{3}{2\sqrt{2}} & -\frac{6}{5} & 0 \\ \frac{\sqrt{5}}{4} & \frac{3}{2\sqrt{2}} & -\frac{17}{12} & -\sqrt{2} & 0 \\ -\sqrt{\frac{2}{5}} & -\frac{6}{5} & -\sqrt{2} & -\frac{16}{15} & 0 \\ 0 & 0 & 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad \lambda_1 \lambda_6 = \begin{pmatrix} -\frac{11}{12} & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{\sqrt{5}}{4} & 0 & 0 \\ \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{13}{10} & -\frac{1}{6\sqrt{2}} & -\frac{4}{15} & 0 \\ -\frac{\sqrt{5}}{4} & -\frac{1}{6\sqrt{2}} & -\frac{37}{36} & -\frac{4\sqrt{2}}{9} & 0 \\ 0 & -\frac{4}{15} & -\frac{4\sqrt{2}}{9} & -\frac{64}{45} & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}, \\
\lambda_5 \lambda_6 = & \begin{pmatrix} -\frac{31}{24} & -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{3\sqrt{5}}{8} & 0 & 0 \\ -\frac{3\sqrt{\frac{5}{2}}}{4} & -\frac{43}{60} & \frac{1}{4\sqrt{2}} & \frac{2}{5} & 0 \\ \frac{3\sqrt{5}}{8} & \frac{1}{4\sqrt{2}} & -\frac{9}{8} & \frac{2\sqrt{2}}{3} & 0 \\ 0 & \frac{2}{5} & \frac{2\sqrt{2}}{3} & -\frac{8}{15} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{F5}
\end{aligned}$$

$$\begin{aligned}
\lambda_1 \lambda_2 \sigma_1 \sigma_2 &= \begin{pmatrix} \frac{7}{6} & \frac{7}{3\sqrt{10}} & -\frac{11\sqrt{5}}{18} & \frac{\sqrt{\frac{2}{5}}}{9} & 0 \\ \frac{7}{3\sqrt{10}} & \frac{3}{5} & -\frac{7}{9\sqrt{2}} & \frac{34}{45} & 0 \\ -\frac{11\sqrt{5}}{18} & -\frac{7}{9\sqrt{2}} & \frac{19}{54} & -\frac{\sqrt{2}}{27} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{9} & \frac{34}{45} & -\frac{\sqrt{2}}{27} & -\frac{16}{135} & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}, \quad \lambda_1 \lambda_4 \sigma_1 \sigma_4 = \begin{pmatrix} \frac{15}{8} & -\frac{13}{12\sqrt{10}} & \frac{23\sqrt{5}}{72} & -\frac{\sqrt{\frac{2}{5}}}{9} & 0 \\ -\frac{13}{12\sqrt{10}} & \frac{31}{36} & \frac{17}{36\sqrt{2}} & -\frac{4}{9} & 0 \\ \frac{23\sqrt{5}}{72} & \frac{17}{36\sqrt{2}} & -\frac{71}{216} & -\frac{\sqrt{2}}{27} & 0 \\ -\frac{\sqrt{\frac{2}{5}}}{9} & -\frac{4}{9} & -\frac{\sqrt{2}}{27} & -\frac{56}{27} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \\
\lambda_4 \lambda_5 \sigma_4 \sigma_5 &= \begin{pmatrix} -\frac{11}{4} & -\frac{1}{2\sqrt{10}} & -\frac{\sqrt{5}}{12} & \frac{\sqrt{\frac{2}{5}}}{3} & 0 \\ -\frac{1}{2\sqrt{10}} & -\frac{89}{30} & -\frac{1}{2\sqrt{2}} & \frac{2}{5} & 0 \\ -\frac{\sqrt{5}}{12} & -\frac{1}{2\sqrt{2}} & -\frac{37}{12} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{\frac{2}{5}}}{3} & \frac{2}{5} & \frac{\sqrt{2}}{3} & -\frac{16}{5} & 0 \\ 0 & 0 & 0 & 0 & -\frac{8}{3} \end{pmatrix}, \quad \lambda_1 \lambda_6 \sigma_1 \sigma_6 = \begin{pmatrix} -\frac{19}{12} & \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{7}{12\sqrt{5}} & 0 & \frac{4}{3\sqrt{5}} \\ \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{18} & -\frac{47}{90\sqrt{2}} & \frac{4}{9} & \frac{8\sqrt{2}}{15} \\ -\frac{7}{12\sqrt{5}} & -\frac{47}{90\sqrt{2}} & -\frac{307}{540} & -\frac{28\sqrt{2}}{135} & -\frac{92}{45} \\ 0 & \frac{4}{9} & -\frac{28\sqrt{2}}{135} & -\frac{16}{27} & -\frac{32\sqrt{2}}{45} \\ \frac{4}{3\sqrt{5}} & \frac{8\sqrt{2}}{15} & -\frac{92}{45} & -\frac{32\sqrt{2}}{45} & \frac{14}{5} \end{pmatrix}, \\
\lambda_5 \lambda_6 \sigma_5 \sigma_6 &= \begin{pmatrix} -\frac{13}{8} & -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{7}{8\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ -\frac{3\sqrt{\frac{5}{2}}}{4} & \frac{1}{12} & \frac{47}{60\sqrt{2}} & -\frac{2}{3} & -\frac{4\sqrt{2}}{5} \\ \frac{7}{8\sqrt{5}} & \frac{47}{60\sqrt{2}} & -\frac{557}{360} & \frac{14\sqrt{2}}{45} & -\frac{26}{15} \\ 0 & -\frac{2}{3} & \frac{14\sqrt{2}}{45} & \frac{8}{9} & \frac{16\sqrt{2}}{15} \\ -\frac{2}{\sqrt{5}} & -\frac{4\sqrt{2}}{5} & -\frac{26}{15} & \frac{16\sqrt{2}}{15} & -\frac{7}{15} \end{pmatrix}. \tag{F6}
\end{aligned}$$

4. S=3

$$\lambda_1 \lambda_2 = -\frac{2}{3}, \quad \lambda_1 \lambda_4 = -1, \quad \lambda_4 \lambda_5 = -\frac{8}{3}, \quad \lambda_1 \lambda_6 = -2, \quad \lambda_5 \lambda_6 = \frac{1}{3}. \tag{F7}$$

$$\lambda_1 \lambda_2 \sigma_1 \sigma_2 = -\frac{2}{3}, \quad \lambda_1 \lambda_4 \sigma_1 \sigma_4 = -1, \quad \lambda_4 \lambda_5 \sigma_4 \sigma_5 = -\frac{8}{3}, \quad \lambda_1 \lambda_6 \sigma_1 \sigma_6 = -2, \quad \lambda_5 \lambda_6 \sigma_5 \sigma_6 = \frac{1}{3}. \tag{F8}$$

For each spin, $\sum_{i<j}^6 \lambda_i \lambda_j = -16I_n$ where I_n is the $n \times n$ identity matrix.

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